Report CS 00–06

Mathematical Analysis of the DtN Finite Element Method for the Exterior Helmholtz Problem

Daisuke KOYAMA

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Abstract

The DtN finite element method for solving the exterior Helmholtz problem is mathematically analyzed. The reduced problem with DtN artificial boundary condition is shown to be equivalent to the original exterior problem. Error estimates for solutions obtained by the DtN finite element method are established.

1 Introduction

We consider the exterior Helmholtz problem with the outgoing radiation condition:

$$
(1) \quad \begin{cases} -\Delta u - k^2 u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \gamma, \\ \displaystyle\lim_{r \longrightarrow +\infty} r^{\frac{d-1}{2}} \left(\frac{du}{dr} - iku \right) &= 0. \end{cases}
$$

Here k, called wave number, is a positive constant, Ω is an unbounded domain of $\boldsymbol{R}^d$ ($d = 2$ or 3) with sufficiently smooth boundary γ, and $r = |x|$ for $x \in \boldsymbol{R}^d$. We assume that $\mathcal{O} = \boldsymbol{R}^d \setminus \overline{\Omega}$ is a bounded open set and that f has a compact support.

When numerically solving problem (1), one often introduces an artificial boundary in order to reduce the computational domain to a bounded domain and imposes an *artificial boundary condition* (ABC) on the artificial boundary. Imposing the *Drichlet-to-Neumann* (DtN) ABC, we can reduce problem (1) equivalently to a problem on the bounded domain between the boundary γ and the artificial boundary. The DtN ABC is given in the following form: on the artificial boundary Γ_a,

$$
\frac{\partial u}{\partial n} = -\mathcal{S} u,
$$

where n is the unit normal vector on Γ_a being toward infinity and $\mathcal{S}$ is the DtN operator for the Helmholtz equation with the outgoing radiation condition. We choose the artificial boundary Γ_a as follows: $\Gamma_a = \{x \in \boldsymbol{R}^d \mid |x| = a\}$, where a is a positive number such that $\overline{\mathcal{O}} \cup \operatorname{supp} f \subset \{x \in \boldsymbol{R}^d \mid |x| < a\}$. Then the bounded computational domain is defined by $\Omega_a = \{x \in \Omega \mid |x| < a\}$ (see Fig. 1), and further the reduced problem is as follows:

$$
(2) \quad \begin{cases} -\Delta u - k^2 u &= f \quad \text{in } \Omega_a, \\ u &= 0 \quad \text{on } \gamma, \\ \dfrac{\partial u}{\partial n} &= -\mathcal{S} u \quad \text{on } \Gamma_a. \end{cases}
$$

We discretize problem (2) by the finite element method in order to compute numerical solutions. The obtained discrete problem can not be computed because the DtN operator is analytically represented with an infinite series. Hence this infinite series has to be truncated in practice.

Our main goals are to show that problem (2) is equivalent to problem (1) and to establish error estimates for solutions of the discrete problems with and without the truncation of the DtN operator.

The remainder of this report is organized as follows. In Section 2, we show well-posedness of problem (1). In Section 3, we define the DtN operator by using the Hankel functions, properties of which are studied in Section 4 and are used in the following sections. In Section 5, we show well-posedness of problem (2). In Section 6, we show the equivalence between problems (1) and (2). In Section 7, we establish the error estimates mentioned above.

2 Uniqueness and existence of the solution to the exterior Helmholtz problem

We define, for every domain $\Omega \subset \boldsymbol{R}^d$,

$$
L^2_{\text{loc}}(\overline{\Omega}) = \{u \mid u \in L^2(B) \ \text{ for all bounded open set } B \subset \Omega\},
$$

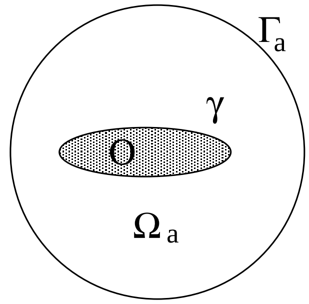

Figure 1: Artificial boundary Γ_a and computational domain Ω_a.

$$H^m_{\rm loc}(\overline{\Omega}) \;=\; \{u \mid u \in H^m(B) \;\text{ for all bounded open set } B \subset \Omega\} \quad (m \in \boldsymbol{N}).$$

THEOREM **2.1** *For every compactly supported $f \in L^2(\Omega)$, problem* (1) *has a unique solution in $H^2_{\rm loc}(\overline{\Omega})$.*

We prove Theorem 2.1 by following Phillips [10] and Sanchez Hubert and Sanchez Palencia [11]. To do so, we present several lemmas and a proposition in the following.

We here denote by ψ the fundamental solution of the Helmholtz equation which satisfies the outgoing radiation condition:

$$(3) \qquad \psi(x) = \begin{cases} \dfrac{i}{4} H_0^{(1)}(k|x|) & \text{if} \quad d = 2, \\ \dfrac{1}{4\pi}\dfrac{e^{ik|x|}}{|x|} & \text{if} \quad d = 3, \end{cases}$$

where $H_0^{(1)}$ is the cylindrical Hankel function of the first kind of order zero.

LEMMA **2.1** *Let f be a function of $L^2(\boldsymbol{R}^d)$ with compact support. Define*

$$u = \psi * f.$$

Then we have $u \in L^2_{\rm loc}(\boldsymbol{R}^d)$ and

$$-\Delta u - k^2 u = f \quad \text{in} \;\; \boldsymbol{R}^d$$

in the sense of the distribution.

Proof. We first consider the case of $d = 3$. Let B be a bounded measurable set of $\boldsymbol{R}^d$. Set $K = \operatorname{supp} f$. Noting that $\psi \in L^2_{\rm loc}(\boldsymbol{R}^3)$, we have

$$\begin{aligned} \int_B |u(x)|^2\,dx &= \int_B \Big| \int_{\boldsymbol{R}^3} f(y)\psi(x-y)\,dy\Big|^2\,dx \\ &= \int_B \Big| \int_K f(y)\psi(x-y)\,dy\Big|^2\,dx \\ &\le \|f\|^2_{L^2(K)} \int_B\int_K |\psi(x-y)|^2\,dydx. \end{aligned}$$

There exists an $R > 0$ such that $B,\ K \subset U_R$, where $U_R = \{x \in \boldsymbol{R}^3 \mid |x| < R\}$. For every $x \in B$ and for every $y \in K$, we have $|x-y| \le |x| + |y| < 2R$. Hence we have

$$\int_B\int_K |\psi(x-y)|^2\,dydx \le ({\rm meas}B)\|\psi\|^2_{L^2(U_{2R})}.$$

Therefore we have

$$\int_B |u(x)|^2\,dx < +\infty.$$

In the case of $d = 2$, we can prove in exactly the same way as above, since $\psi \in L^2_{\mathrm{loc}}(\boldsymbol{R}^2)$, which follows from the following asymptotic behavior:

$$H_0^{(1)}(kr) \sim i\frac{2}{\pi}\log r$$

for $r \longrightarrow 0$ (see Abramowitz and Stegun [1]). ∎

LEMMA **2.2** *Let f be a function of $L^2(\Omega)$ such that* $\operatorname{supp} f \subset \overline{\Omega_a}$. *Let $u \in H^1_{\mathrm{loc}}(\overline{\Omega})$ be a solution to problem (1). Then u belongs to $C^\infty(\Omega'_a)$ with $\Omega'_a = \left\{x \in \boldsymbol{R}^d \mid |x| > a\right\}$ and can be analytically represented as follows. In the two dimensional case,*

$$(4)\qquad u(r,\,\theta) = \sum_{n=-\infty}^{\infty} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} u_n(a) Y_n(\theta),$$

where r, θ are the polar coordinates, $H_n^{(1)}$ are the cylindrical Hankel functions of the first kind of order n, Y_n are the spherical harmonics defined by

$$Y_n(\theta) = \frac{e^{in\theta}}{\sqrt{2\pi}},$$

and $u_n(r)$ $(r \geq a)$ are the Fourier coefficients defined by

$$(5)\qquad u_n(r) = \int_0^{2\pi} u(r,\,\theta)\overline{Y_n(\theta)}\,d\theta.$$

In the three dimensional case,

$$(6)\qquad u(r,\,\theta,\,\phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} u_n^m(a) Y_n^m(\theta,\,\phi),$$

where r, θ, ϕ are the spherical coordinates, $h_n^{(1)}$ are the spherical Hankel functions of the first kind of order n, Y_n^m are the spherical harmonics defined by

$$Y_n^m(\theta,\,\phi) = \sqrt{\frac{(2n+1)}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)e^{im\phi},$$

where P_n^m are the associated Legendre functions, and $u_n^m(r)$ $(r \geq a)$ are the Fourier coefficients defined by

$$u_n^m(r) = \int_0^{2\pi}\int_0^{\pi} u(r,\,\theta,\,\phi)\overline{Y_n^m(\theta,\,\phi)}\sin\theta\,d\theta d\phi.$$

Proof. By the usual interior regularity theory, we have $u \in C^\infty(\Omega'_a)$.
We next show (4). For every $r > a$, we have

$$u(r,\,\theta) = \sum_{n=-\infty}^{\infty} u_n(r)Y_n(\theta) \quad \text{in } L^2(\Gamma_r),$$

where $\Gamma_r = \left\{x \in \boldsymbol{R}^d \mid |x| = r\right\}$. Then the Fourier coefficients $u_n \in C^\infty((a, \infty))$. Since

$$-\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - k^2 u = 0 \quad \text{in } \Omega_a',$$

we have, for $r > a$,

$$\begin{aligned} 0 &= \int_0^{2\pi} \left(-\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - k^2 u\right) \frac{e^{-in\theta}}{\sqrt{2\pi}}\, d\theta \\ &= -u_n''(r) - \frac{1}{r}u_n'(r) + \left(\frac{n^2}{r^2} - k^2\right) u_n(r). \end{aligned}$$

Hence, u_n can be represented as follows:

$$(7) \quad u_n(r) = A_n H_n^{(1)}(kr) + B_n H_n^{(2)}(kr),$$

where A_n and B_n are constants and $H_n^{(2)}$ are the cylindrical Hankel functions of the second kind of order n. We here note the asymptotic behavior, as r tends to infinity, of the Hankel functions of order $\nu \in \boldsymbol{R}$:

$$(8) \quad H_\nu^{(j)}(kr) = \sqrt{\frac{2}{\pi kr}} e^{\pm i(kr - \frac{\pi}{2}\nu - \frac{\pi}{4})} + O(r^{-3/2}),$$

with sign $+$ and $-$ for $j = 1$ and 2, respectively [1], and hence, by setting $\nu = n + 1$ in (8), we also have

$$(9) \quad H_{n+1}^{(j)}(kr) = \mp i \sqrt{\frac{2}{\pi kr}} e^{\pm i(kr - \frac{\pi}{2}n - \frac{\pi}{4})} + O(r^{-3/2}).$$

We here have, for $j = 1, 2$,

$$(10) \quad \left(\frac{d}{dr} - ik\right) H_n^{(j)}(kr) = \frac{n}{r} H_n^{(j)}(kr) - kH_{n+1}^{(j)}(kr) - ikH_n^{(j)}(kr)$$

because we have the following recurrence relations [1]:

$$H_n^{(j)\prime}(x) = \frac{n}{x} H_n^{(j)}(x) - H_{n+1}^{(j)}(x).$$

Combining (8) with $\nu = n$, (9), and (10), we can get

$$\begin{aligned} (11) \quad \left(\frac{d}{dr} - ik\right) H_n^{(1)}(kr) &= O(r^{-3/2}), \\ (12) \quad \left(\frac{d}{dr} - ik\right) H_n^{(2)}(kr) &= -2i\sqrt{\frac{2k}{\pi r}} e^{-i(kr - \frac{\pi}{2}n - \frac{\pi}{4})} + O(r^{-3/2}). \end{aligned}$$

Since u satisfies the outgoing radiation condition, we have

$$\left\|\frac{\partial u}{\partial r} - iku\right\|_{L^2(\Gamma_r)}^2 = \int_0^{2\pi} \left|\frac{\partial u}{\partial r} - iku\right|^2 r\, d\theta \longrightarrow 0 \quad (r \longrightarrow \infty).$$

This implies that for all $n \in \boldsymbol{Z}$,

$$(13) \quad \sqrt{r}(u_n'(r) - iku_n(r)) \longrightarrow 0 \quad (r \longrightarrow \infty)$$

because

$$\left\| \frac{\partial u}{\partial r} - iku \right\|^2_{L^2(\Gamma_r)} = \sum_{n=-\infty}^{\infty} \left| \sqrt{r}(u_n'(r) - iku_n(r)) \right|^2 .$$

From (7), (11), (12), and (13), we can deduce

$$B_n = 0 \quad \text{for all } n \in \boldsymbol{Z}.$$

We next show

$$(14) \quad A_n = \frac{1}{H_n^{(1)}(ka)} u_n(a).$$

Since $u \in H^1_{\mathrm{loc}}(\overline{\Omega'_a})$, we have

$$(15) \quad u(r,\, \theta) \longrightarrow u(a,\, \theta) \quad \text{in } L^2(0,\, 2\pi) \quad (r \longrightarrow a+0).$$

This can be shown by using the trace theorem and the fact that $C^\infty(\overline{\Omega^b_a})$ is dense in $H^1(\Omega^b_a)$ for every $b > a$, where

$$(16) \quad \begin{aligned} \Omega^b_a &= \{x \in \boldsymbol{R}^d \mid a < |x| < b\}, \\ C^\infty(\overline{\Omega^b_a}) &= \left\{ u = \tilde{u}|_{\Omega^b_a} \mid \tilde{u} \in C^\infty(\boldsymbol{R}^d) \right\}. \end{aligned}$$

It is easily seen from (15) that

$$(17) \quad u_n(r) \longrightarrow u_n(a) \quad (r \longrightarrow a+0).$$

On the other hand, we have

$$(18) \quad u_n(r) = A_n H_n^{(1)}(kr) \longrightarrow A_n H_n^{(1)}(ka) \quad (r \longrightarrow a+0).$$

From (17) and (18), we have

$$u_n(a) = A_n H_n^{(1)}(ka),$$

which implies (14). Therefore we get (4).

Next we consider the case of $d = 3$. Then u ($\in C^\infty(\Omega'_a)$) can be represented, for every $r > a$, as follows:

$$u(r,\, \theta,\, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} u_n^m(r) Y_n^m(\theta,\, \phi) \quad \text{in } L^2(\Gamma_r).$$

Then Fourier coefficients u_n^m satisfy

$$-\frac{d^2 u_n^m}{dr^2}(r) - \frac{2}{r}\frac{du_n^m}{dr}(r) + \left(\frac{n(n+1)}{r^2} - k^2 \right) u_n^m(r) = 0,$$

and hence

$$u_n^m(r) = A_n^m h_n^{(1)}(kr) + B_n^m h_n^{(2)}(kr),$$

where $h_n^{(2)}$ are the spherical Hankel functions of the second kind of order n. From (8) and

$$(19) \quad h_n^{(j)}(x) = \sqrt{\frac{\pi}{2x}} H^{(j)}_{n+1/2}(x)$$

($j = 1$, 2 and $n \in \boldsymbol{N} \cup \{0\}$), we can get the asymptotic behavior of $h_n^{(j)}$:

$$(20)\quad h_n^{(j)}(kr) = \frac{(\mp i)^{n+1}}{kr} e^{\pm ikr} + O(r^{-2}) \quad (r \longrightarrow +\infty),$$

with sign $+$ and $-$ for $j = 1$ and 2, respectively. We further have the following recurrence relations [1]:

$$(21)\quad h_n^{(j)\prime}(x) = \frac{n}{x} h_n^{(j)}(x) - h_{n+1}^{(j)}(x) \quad (j = 1,\, 2).$$

Using (20) and (21), we can show

$$A_n^m = \frac{1}{h_n^{(1)}(ka)} u_n^m(a), \quad B_n^m = 0$$

in the same way as in the case of $d = 2$. Thus we can get (6). ■

Lemma **2.3** *Let f be a function of $L^2(\Omega)$ such that* $\operatorname{supp} f \subset \overline{\Omega_a}$. *Let $u \in H^1_{\mathrm{loc}}(\overline{\Omega})$ be a solution to problem* (1). *Then, there exists a $\Phi \geq 0$ such that*

$$\Phi = \int_{\Gamma_r} \frac{\partial u}{\partial n} \overline{u}\, d\gamma \quad \textit{for all } r > a,$$

and further

$$(22)\quad \Phi = \lim_{r \longrightarrow \infty} \frac{1}{2k} \int_{\Gamma_r} \left[\left| \frac{\partial u}{\partial n} \right|^2 + k^2 |u|^2 \right] d\gamma,$$

where n is the unit normal vector being toward infinity.

Proof. Since $u \in C^\infty(\Omega'_a)$, by the Green formula, we have, for $r' > r > a$,

$$\begin{aligned} 0 &= -\int_{\Omega_r^{r'}} (\Delta u + k^2 u)\overline{u}\, dx \\ &= \int_{\Gamma_r} \frac{\partial u}{\partial n} \overline{u}\, d\gamma - \int_{\Gamma_{r'}} \frac{\partial u}{\partial n} \overline{u}\, d\gamma + \int_{\Omega_r^{r'}} (|\nabla u|^2 - k^2 |u|^2)\, dx, \end{aligned}$$

where $\Omega_r^{r'}$ is the annular domain defined by (16). Taking the imaginary part of this identity, we get

$$\operatorname{Im} \int_{\Gamma_r} \frac{\partial u}{\partial n} \overline{u}\, d\gamma = \operatorname{Im} \int_{\Gamma_{r'}} \frac{\partial u}{\partial n} \overline{u}\, d\gamma \equiv \Phi.$$

Here we note that for $r > a$,

$$\begin{aligned} \int_{\Gamma_r} \left| \frac{\partial u}{\partial n} - iku \right|^2 d\gamma &= \int_{\Gamma_r} \left(\frac{\partial u}{\partial n} - iku \right) \left(\frac{\partial \overline{u}}{\partial n} + ik\overline{u} \right) d\gamma \\ &= \int_{\Gamma_r} \left[\left| \frac{\partial u}{\partial n} \right|^2 - iku \frac{\partial \overline{u}}{\partial n} + ik \frac{\partial u}{\partial n} \overline{u} + k^2 |u|^2 \right] d\gamma \\ &= \int_{\Gamma_r} \left[\left| \frac{\partial u}{\partial n} \right|^2 + k^2 |u|^2 \right] d\gamma + ik \int_{\Gamma_r} \left(\frac{\partial u}{\partial n} \overline{u} - u \frac{\partial \overline{u}}{\partial n} \right) d\gamma. \end{aligned}$$

From this identity, we get

$$\begin{aligned}\operatorname{Im}\int_{\Gamma_r}\frac{\partial u}{\partial n}\overline{u}\,d\gamma &\equiv \frac{1}{2i}\int_{\Gamma_r}\left(\frac{\partial u}{\partial n}\overline{u}-u\frac{\partial \overline{u}}{\partial n}\right)d\gamma\\ &= \frac{1}{2k}\left\{\int_{\Gamma_r}\left[\left|\frac{\partial u}{\partial n}\right|^2+k^2|u|^2\right]d\gamma-\int_{\Gamma_r}\left|\frac{\partial u}{\partial n}-iku\right|^2d\gamma\right\}.\end{aligned}$$

Letting $r \longrightarrow +\infty$ in this identity, we obtain (22) since u satisfies the outgoing radiation condition. ■

LEMMA **2.4** *Let f be a function of $L^2(\Omega)$ such that* $\operatorname{supp} f \subset \overline{\Omega_a}$. *Let $u \in H^1_{\mathrm{loc}}(\overline{\Omega})$ be a solution to problem* (1). *Suppose that*

$$(23)\quad \lim_{r\longrightarrow+\infty}\int_{\Gamma_r}|u|^2\,d\gamma = 0,$$

then we have

$$u(x)\equiv 0 \quad \textit{for } |x|\geq a.$$

Proof. We prove only the two dimensional case. For the three dimensional case, we can prove analogously. By Lemma 2.2, we have for every $r \geq a$,

$$u(r,\theta)=\sum_{n=-\infty}^{\infty}\frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)}u_n(a)Y_n(\theta)\quad \text{in } L^2(\Gamma_r),$$

and hence we have

$$\int_{\Gamma_r}|u|^2\,d\gamma=\sum_{n=-\infty}^{\infty}\left|\sqrt{r}\frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)}u_n(a)\right|^2.$$

It follows from (23) that for all $n \in \boldsymbol{Z}$,

$$(24)\quad \lim_{r\longrightarrow+\infty}\left|\sqrt{r}H_n^{(1)}(kr)u_n(a)\right|=0.$$

Combining (24) and (8) deduces that $u_n(a)=0$ for all $n \in \boldsymbol{Z}$. This implies $u \equiv 0$ for $r \geq a$. ■

PROPOSITION **2.1** *For $f \in L^2(\Omega)$ with compact support, a solution to problem* (1) *which belongs to $H^1_{\mathrm{loc}}(\overline{\Omega})$ is unique.*

Proof. Let $u \in H^1_{\mathrm{loc}}(\overline{\Omega})$ be a solution to problem (1) with $f = 0$. Then we can see from the usual interior regularity argument that $u \in C^\infty(\Omega)$.

We first show

$$(25)\quad \operatorname{Im}\int_{\Gamma_r}\frac{\partial u}{\partial n}\overline{u}\,d\gamma = 0$$

for all $r > a$. Define

$$C_0^\infty(\Omega_r\cup\Gamma_r)=\{u=\tilde{u}|_{\Omega_r}\mid \tilde{u}\in C_0^\infty(\Omega)\}$$

with $\Omega_r = \{x \in \Omega \mid |x| < r\}$. We here fix $r > a$. There exist $u_j \in C_0^\infty(\Omega_r \cup \Gamma_r)$ ($j = 1, 2, \ldots$) such that

$$u_j \longrightarrow u \quad \text{in } H^1(\Omega_r) \quad (j \longrightarrow \infty).$$

For every $j \in \boldsymbol{N}$, we have

$$\begin{aligned} 0 &= -\int_{\Omega_r} (\Delta u - k^2 u)\overline{u_j}\, dx \\ &= -\int_{\Gamma_r} \frac{\partial u}{\partial n}\overline{u_j}\, d\gamma + \int_{\Omega_r} (\nabla u \cdot \nabla \overline{u_j} - k^2 u \overline{u_j})\, dx. \end{aligned}$$

Letting $j \longrightarrow \infty$ in this identity, we get

$$-\int_{\Gamma_r} \frac{\partial u}{\partial n}\overline{u}\, d\gamma + \int_{\Omega_r} (|\nabla u|^2 - k^2|u|^2)\, dx = 0.$$

Taking the imaginary part of this identity, we obtain (25).

From (25) and Lemma 2.3, we can get

$$\lim_{r \longrightarrow \infty} \frac{1}{2k} \int_{\Gamma_r} \left[\left| \frac{\partial u}{\partial n} \right|^2 + k^2 |u|^2 \right] d\gamma = 0.$$

This implies

$$\lim_{r \longrightarrow \infty} \int_{\Gamma_r} |u|^2\, d\gamma = 0.$$

Hence, by Lemma 2.4, we have

$$u \equiv 0 \quad \text{for } |x| \geq a.$$

By virtue of the unique continuation property, we have $u \equiv 0$ in Ω. ■

Now we present proof of Theorem 2.1.

Proof of Theorem 2.1. Uniqueness of the solution to problem (1) follows from Proposition 2.1. So we will show existence of the solution in the following.

Let g be any function of $L^2(\Omega_a)$. We shall extend g by zero on $\boldsymbol{R}^d \setminus \Omega_a$ and denote the extension by the same symbol g, and thus $g \in L^2(R^d)$. Let us construct

$$w = g * \psi, \tag{26}$$

where ψ denotes the outgoing fundamental solution of the Helmholtz equation, i.e., ψ is a function of $\boldsymbol{R}^d$ defined by (3). Then, we have $w \in H^2_{\mathrm{loc}}(\boldsymbol{R}^d)$ and

$$-\Delta w - k^2 w = g \quad \text{on } \boldsymbol{R}^d$$

in the sense of the distribution. Now, we consider the following problem:

$$\left\{ \begin{array}{rcll} -\Delta v + \mu v &=& 0 & \text{in } \Omega_a, \\ v &=& w & \text{on } \gamma, \\ v &=& 0 & \text{on } \Gamma_a, \end{array} \right. \tag{27}$$

where $\mu \in \boldsymbol{C}$. If $\operatorname{Im}\mu \neq 0$, this problem has a unique solution $v \in H^2(\Omega_a)$, since boundaries γ and Γ_a are sufficiently smooth. We choose $\phi \in C^\infty(\overline{\Omega_a})$ to be identically one in a neighborhood of γ and identically zero in a neighborhood of Γ_a. We then seek u defined on Ω under the form

$$(28)\quad u = w - \phi v.$$

We here note that $u \in H^2_{\mathrm{loc}}(\overline{\Omega})$ because $w \in H^2_{\mathrm{loc}}(\boldsymbol{R}^d)$ and $\phi v \in H^2(\Omega)$. Then $u = 0$ on γ, and u satisfies the outgoing radiation condition. In order to satisfy the Helmholtz equation, the following relation between f and g must hold on Ω:

$$(29)\quad \begin{aligned} f &= -(\Delta + k^2)u \\ &= g + (\Delta\phi + k^2\phi + \mu\phi)v + 2\nabla\phi \cdot \nabla v, \end{aligned}$$

and we note that (29) takes the form $0 = 0$ for $|x| > a$. Consequently, (29) must be considered as a condition on Ω_a. We shall write it in the form

$$(30)\quad f = g + Kg \quad \text{in } L^2(\Omega_a).$$

We show below that for every $f \in L^2(\Omega_a)$, there exists a $g \in L^2(\Omega_a)$ satisfying (30). For this purpose, we first show $K : \ L^2(\Omega_a) \longrightarrow L^2(\Omega_a)$ is compact operator in Steps 1–3 below.

Step 1. In this step, we show that the operator $g \longrightarrow w|_{\Omega_a}(\equiv g * \psi|_{\Omega_a})$ belongs to $\mathcal{L}(L^2(\Omega_a),\, H^2(\Omega_a))$, where $\mathcal{L}(L^2(\Omega_a),\, H^2(\Omega_a))$ denotes the set of all bounded linear operators from $L^2(\Omega_a)$ to $H^2(\Omega_a)$.

Take $g_j \in L^2(\Omega_a)$ ($j \in \boldsymbol{N}$) such that $g_j \longrightarrow g$ in $L^2(\Omega_a)$. Set $w_j = g_j * \psi$, where g_j are assumed to be extended by zero on $\boldsymbol{R}^d \setminus \Omega_a$, and suppose

$$w_j|_{\Omega_a} \longrightarrow \widetilde{w} \quad \text{in } H^2(\Omega_a).$$

Then we have

$$\begin{aligned} \|w_j - w\|_{L^2(\Omega_a)} &= \int_{\Omega_a} |\int_{\boldsymbol{R}^d}[g_j(y) - g(y)]\psi(x-y)\,dy|^2\,dx \\ &= \int_{\Omega_a} |\int_{\Omega_a}[g_j(y) - g(y)]\psi(x-y)\,dy|^2\,dx \\ &= \int_{\Omega_a} |g_j(y) - g(y)|^2\,dy \int_{\Omega_a}\int_{\Omega_a} |\psi(x-y)|^2\,dydx \\ &\longrightarrow 0 \quad (j \longrightarrow \infty). \end{aligned}$$

This implies $\widetilde{w} = w|_{\Omega_a}$. Hence, the operator $g \longrightarrow w|_{\Omega_a}$ is a closed operator from $L^2(\Omega_a)$ into $H^2(\Omega_a)$, and hence we can see from the closed graph theorem that the operator $g \longrightarrow w|_{\Omega_a}$ belongs to $\mathcal{L}(L^2(\Omega_a),\, H^2(\Omega_a))$.

Step 2. In this step, we show that the operator $w \longrightarrow v$ belongs to $\mathcal{L}(H^2(\Omega_a)) \equiv \mathcal{L}(H^2(\Omega_a),\, H^2(\Omega_a))$.

It follows from the trace theorem that

$$(31)\quad w \longrightarrow w|_\gamma \ \in \mathcal{L}(H^2(\Omega_a),\, H^{3/2}(\gamma)).$$

Further, we see from the regularity theorem and the closed graph theorem that

$$(32)\quad w|_\gamma \longrightarrow v \ \in \mathcal{L}(H^{3/2}(\gamma),\, H^2(\Omega_a)).$$

From (31) and (32) it follows that

$$w \longrightarrow v \ \in \mathcal{L}(H^2(\Omega_a)).$$

Step 3. The operator

$$v \longrightarrow (\Delta\phi + k^2\phi + \mu\phi)v + 2\nabla\phi \cdot \nabla v$$

is a compact operator from $H^2(\Omega_a)$ into $L^2(\Omega_a)$. This follows from the compact imbedding of $H^2(\Omega_a)$ into $H^1(\Omega_a)$.

From Steps 1–3, we can see K is a compact operator on $L^2(\Omega_a)$.

Since K is a compact operator on $L^2(\Omega_a)$, it suffices to show $I+K$ is one-to-one in order to show that the equation (30) has a unique solution $g \in L^2(\Omega_a)$ for every $f \in L^2(\Omega_a)$. We assume that $g \in L^2(\Omega_a)$ satisfies

$$g + Kg = 0.$$

Using g, we construct u by (26), (27), and (28). Then $u \in H^1_{\mathrm{loc}}(\overline{\Omega})$ and

$$-(\Delta + k^2)u = 0 \quad \text{in } \Omega.$$

Thus, by Proposition 2.1, $u = 0$ in Ω. Here we get

$$w = \phi v \quad \text{in } \Omega. \tag{33}$$

Since v is the solution of (27), we have

$$\int_{\Omega_a} (|\nabla v|^2 + \mu|v|^2)\, dx = \int_\gamma \frac{\partial v}{\partial n} \overline{v}\, d\gamma, \tag{34}$$

where n is the outer unit normal vector to Ω_a.

On the other hand, since $-\Delta w - k^2 w = g \equiv 0$ in $\mathcal{O}$, we have

$$\int_{\mathcal{O}} (|\nabla w|^2 - k^2|w|^2)\, dx = -\int_\gamma \frac{\partial w}{\partial n} \overline{w}\, d\gamma. \tag{35}$$

Now, adding (34) and (35), and because of the fact that $w = v$ in a neighborhood on γ, we see that the right-hand sides of (34) and (35) cancel. Further, taking the imaginary part, we obtain

$$(\mathrm{Im}\,\mu) \int_{\Omega_a} |v|^2\, dx = 0.$$

Thus we have $v = 0$ in Ω_a, and hence, by (33), $w = 0$ in Ω. Therefore, since

$$-(\Delta + k^2)w = g \quad \text{in } \boldsymbol{R}^d,$$

we have $g = 0$. This implies $I + K$ is one-to-one. ■

3 DtN operator

We define the DtN operator $\mathcal{S}$ as follows. In the two dimensional case, for $\varphi \in H^{1/2}(\Gamma_a)$,

$$\mathcal{S}\varphi = \sum_{n=-\infty}^{\infty} -k \frac{H_n^{(1)'}(ka)}{H_n^{(1)}(ka)} \varphi_n Y_n,$$

where

$$\varphi_n = \int_0^{2\pi} \varphi(\theta) \overline{Y_n(\theta)}\, d\theta. \tag{36}$$

In the three dimensional case, for $\varphi \in H^{1/2}(\Gamma_a)$,

$$\mathcal{S}\varphi = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} -k \frac{h_n^{(1)\prime}(ka)}{h_n^{(1)}(ka)} \varphi_n^m Y_n^m,$$

where

$$(37) \quad \varphi_n^m = \int_0^{2\pi} \int_0^{\pi} \varphi(\theta,\, \phi) \overline{Y_n^m(\theta,\, \phi)} \sin\theta \, d\theta d\phi.$$

The DtN operator $\mathcal{S}$ is a bounded linear operator from $H^{1/2}(\Gamma_a)$ into $H^{-1/2}(\Gamma_a)$ (see Masmoudi [9], Koyama [8]).

4 Properties of the Hankel functions

Lemma **4.1** *For each $x > 0$, we have*

$$(38) \quad H_\nu^{(1)}(x) \sim -i\sqrt{\frac{2}{\pi\nu}} \left(\frac{ex}{2\nu}\right)^{-\nu} \quad (\nu \longrightarrow \infty),$$

where $\nu \in \boldsymbol{R}$.

Proof. According to [1], we have

$$(39) \quad J_\nu(x) \sim \frac{1}{\sqrt{2\pi\nu}} \left(\frac{ex}{2\nu}\right)^{\nu} \quad (\nu \longrightarrow \infty),$$

$$(40) \quad N_\nu(x) \sim -\sqrt{\frac{2}{\pi\nu}} \left(\frac{ex}{2\nu}\right)^{-\nu} \quad (\nu \longrightarrow \infty),$$

where J_ν and N_ν are the cylindrical Bessel functions and the cylindrical Neumann functions of order ν, respectively. We have

$$\begin{aligned}(41) \quad & H_\nu^{(1)}(x) \left\{-i\sqrt{\frac{2}{\pi\nu}} \left(\frac{ex}{2\nu}\right)^{-\nu}\right\}^{-1} \\ &= J_\nu(x) \left\{\frac{1}{\sqrt{2\pi\nu}} \left(\frac{ex}{2\nu}\right)^{\nu}\right\}^{-1} \left\{\frac{1}{\sqrt{2\pi\nu}} \left(\frac{ex}{2\nu}\right)^{\nu}\right\} \left\{-i\sqrt{\frac{2}{\pi\nu}} \left(\frac{ex}{2\nu}\right)^{-\nu}\right\}^{-1} \\ &\qquad + N_\nu(x) \left\{-\sqrt{\frac{2}{\pi\nu}} \left(\frac{ex}{2\nu}\right)^{-\nu}\right\}^{-1}.\end{aligned}$$

We here note that

$$(42) \quad \left\{\frac{1}{\sqrt{2\pi\nu}} \left(\frac{ex}{2\nu}\right)^{\nu}\right\} \left\{-i\sqrt{\frac{2}{\pi\nu}} \left(\frac{ex}{2\nu}\right)^{-\nu}\right\}^{-1} = \frac{i}{2} \left(\frac{ex}{2\nu}\right)^{2\nu} \longrightarrow 0 \quad (\nu \longrightarrow \infty).$$

Combining (39) – (42), we can get (38). ■

Lemma **4.2** *For each $x > 0$, we have*

$$(43) \quad \frac{H_{\nu-1}^{(1)}(x)}{H_\nu^{(1)}(x)} \sim \frac{x}{2\nu} \quad (\nu \longrightarrow \infty),$$

where $\nu \in \boldsymbol{R}$.

Proof. We have

$$\text{(44)}\quad \frac{H^{(1)}_{\nu-1}(x)}{H^{(1)}_{\nu}(x)}\frac{2\nu}{x} = \frac{H^{(1)}_{\nu-1}(x)}{-i\sqrt{\frac{2}{\pi(\nu-1)}}\left(\frac{ex}{2(\nu-1)}\right)^{-(\nu-1)}} \frac{-i\sqrt{\frac{2}{\pi\nu}}\left(\frac{ex}{2\nu}\right)^{-\nu}}{H^{(1)}_{\nu}(x)} \frac{-i\sqrt{\frac{2}{\pi(\nu-1)}}\left(\frac{ex}{2(\nu-1)}\right)^{-(\nu-1)}}{-i\sqrt{\frac{2}{\pi\nu}}\left(\frac{ex}{2\nu}\right)^{-\nu}} \frac{2\nu}{x}.$$

We here note that

$$\text{(45)}\quad \begin{aligned} &\frac{-i\sqrt{\frac{2}{\pi(\nu-1)}}\left(\frac{ex}{2(\nu-1)}\right)^{-(\nu-1)}}{-i\sqrt{\frac{2}{\pi\nu}}\left(\frac{ex}{2\nu}\right)^{-\nu}} \frac{2\nu}{x} \\ &= \left(1+\frac{1}{\nu-1}\right)^{3/2}\left\{\left(1-\frac{1}{\nu}\right)^{-\nu}\right\}^{-1} e \longrightarrow 1 \quad (\nu \longrightarrow \infty). \end{aligned}$$

From (44), (45), and Lemma 4.1, we can obtain (43). ■

Lemma **4.3** *Let $k>0$ and $a>0$. Then there exists a positive constant C such that*

$$\text{(46)}\quad \left| k\frac{H^{(1)'}_{n}(ka)}{H^{(1)}_{n}(ka)} + \frac{|n|}{a} \right| \le C \qquad \text{for all } n \in \boldsymbol{Z}.$$

Proof. By the recursion formulas [1]:

$$\text{(47)}\quad H^{(1)'}_{\nu}(x) = H^{(1)}_{\nu-1}(x) - \frac{\nu}{x}H^{(1)}_{\nu}(x) \qquad \text{for all } \nu \in \boldsymbol{R},$$

we can get

$$\text{(48)}\quad k\frac{H^{(1)'}_{n}(ka)}{H^{(1)}_{n}(ka)} = k\frac{H^{(1)}_{n-1}(ka)}{H^{(1)}_{n}(ka)} - \frac{n}{a} \qquad \text{for all } n \in \boldsymbol{Z}.$$

From Lemma 4.2, we can see that there exists a positive constant C such that

$$\left| \frac{H^{(1)}_{n-1}(ka)}{H^{(1)}_{n}(ka)} \frac{2n}{ka} \right| \le C \qquad \text{for all } n \in \boldsymbol{N} \cup \{0\}.$$

This implies that

$$\text{(49)}\quad \left| \frac{H^{(1)}_{n-1}(ka)}{H^{(1)}_{n}(ka)} \right| \le C\frac{ka}{2n} \qquad \text{for all } n \in \boldsymbol{N}.$$

Combining (48) and (49), we can get

$$\left| k\frac{H^{(1)'}_{n}(ka)}{H^{(1)}_{n}(ka)} + \frac{n}{a} \right| \le C\frac{k^2 a}{2n} \qquad \text{for all } n \in \boldsymbol{N}.$$

Further, noting that $H^{(1)}_{-n}(ka) = (-1)^n H^{(1)}_{n}(ka)$, we see that (46) holds. ■

Lemma **4.4** *Let $k>0$ and $a>0$. Then there exists a positive constant C such that*

$$\text{(50)}\quad \left| k\frac{h^{(1)'}_{n}(ka)}{h^{(1)}_{n}(ka)} + \frac{n+1}{a} \right| \le C \qquad \text{for all } n \in \boldsymbol{N} \cup \{0\}.$$

Proof. By (19) and the recurrence relations [1]:

$$(51)\quad h_n^{(1)'}(x) = h_{n-1}^{(1)}(x) - \frac{n+1}{x} h_n^{(1)}(x),$$

we can get

$$(52)\quad k\frac{h_n^{(1)'}(ka)}{h_n^{(1)}(ka)} = k\frac{H_{n-1/2}^{(1)}(ka)}{H_{n+1/2}^{(1)}(ka)} - \frac{n+1}{a} \qquad \text{for all } n \in \boldsymbol{N} \cup \{0\}.$$

We can see from (43) that there exists a positive constant C such that

$$(53)\quad \left|\frac{H_{n-1/2}^{(1)}(ka)}{H_{n+1/2}^{(1)}(ka)}\right| \le C\frac{ka}{2n+1} \qquad \text{for all } n \in \boldsymbol{N} \cup \{0\}.$$

From (52) and (53), we can get (50). ■

Lemma **4.5** *Let $k > 0$ and $a > 0$. Then there exists a positive constant C such that*

$$\left|\frac{1}{1+|n|}\frac{H_n^{(1)'}(ka)}{H_n^{(1)}(ka)}\right| \le C \qquad \text{for all } n \in \boldsymbol{Z}.$$

Proof. It is clear from Lemma 4.3. ■

Lemma **4.6** *Let $k > 0$ and $a > 0$. Then there exists a positive constant C such that*

$$\left|\frac{1}{1+n}\frac{h_n^{(1)'}(ka)}{h_n^{(1)}(ka)}\right| \le C \qquad \text{for all } n \in \boldsymbol{N} \cup \{0\}.$$

Proof. It is clear from Lemma 4.4. ■

Lemma **4.7** *For all $x > 0$ and for all $\nu \in \boldsymbol{R}$, we have*

$$\mathrm{Im}\left\{\frac{H_\nu^{(1)'}(x)}{H_\nu^{(1)}(x)}\right\} > 0.$$

Proof. We have the following formulas [1]:

$$(54)\quad J_{\nu-1}(x)N_\nu(x) - J_\nu(x)N_{\nu-1}(x) = -\frac{2}{\pi x}$$

Using (47) and (54), we can get

$$\mathrm{Im}\left\{\frac{H_\nu^{(1)'}(x)}{H_\nu^{(1)}(x)}\right\} = \frac{2}{\pi x}\frac{1}{J_\nu^2(x) + N_\nu^2(x)} > 0. \quad ■$$

Lemma **4.8** *For all $x > 0$ and for all $n \in \boldsymbol{N} \cup \{0\}$, we have*

$$\mathrm{Im}\left\{\frac{h_n^{(1)'}(x)}{h_n^{(1)}(x)}\right\} > 0.$$

Proof. By using (19), we can get

$$\frac{h_n^{(1)\prime}(x)}{h_n^{(1)}(x)} = -\frac{1}{2x} + \frac{H_{n+1/2}^{(1)\prime}(x)}{H_{n+1/2}^{(1)}(x)}, \tag{55}$$

and hence

$$\mathrm{Im}\left\{\frac{h_n^{(1)\prime}(x)}{h_n^{(1)}(x)}\right\} = \mathrm{Im}\left\{\frac{H_{n+1/2}^{(1)\prime}(x)}{H_{n+1/2}^{(1)}(x)}\right\}.$$

Thus, by Lemma 4.7, we have

$$\mathrm{Im}\left\{\frac{h_n^{(1)\prime}(x)}{h_n^{(1)}(x)}\right\} > 0$$

for all $n \in \boldsymbol{N} \cup \{0\}$. ■

LEMMA **4.9** *For all $x > 0$ and for all $\nu \in \boldsymbol{R}$, we have*

$$\mathrm{Re}\left\{\frac{H_\nu^{(1)\prime}(x)}{H_\nu^{(1)}(x)}\right\} < 0.$$

Proof. Since $H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x)$, we have

$$\mathrm{Re}\left\{\frac{H_\nu^{(1)\prime}(x)}{H_\nu^{(1)}(x)}\right\} = \frac{J_\nu(x)J_\nu'(x) + N_\nu(x)N_\nu'(x)}{J_\nu^2(x) + N_\nu^2(x)}. \tag{56}$$

According to Watson [12], we have

$$J_\nu^2(x) + N_\nu^2(x) = \frac{8}{\pi^2}\int_0^\infty K_0(2x\sinh t)\cosh(2\nu t)\,dt, \tag{57}$$

where K_0 is the modified Bessel function of the second kind of order zero. Differentiating (57) with x, we obtain

$$J_\nu(x)J_\nu'(x) + N_\nu(x)N_\nu'(x) = \frac{8}{\pi^2}\int_0^\infty K_0'(2x\sinh t)\sinh t\cosh(2\nu t)\,dt. \tag{58}$$

Now we note that we have the following formula ([1], [12]):

$$K_0(\xi) = \int_0^\infty e^{-\xi\cosh t}\,dt \quad \text{for all } \xi > 0. \tag{59}$$

Differentiating (59) with ξ, we can get

$$K_0'(\xi) = -\int_0^\infty e^{-\xi\cosh t}\cosh t\,dt < 0 \quad \text{for all } \xi > 0. \tag{60}$$

Combining (56), (58), and (60) will complete the proof of Lemma 4.9. ■

LEMMA **4.10** *For all $x > 0$ and for all $n \in \boldsymbol{N} \cup \{0\}$, we have*

$$\mathrm{Re}\left\{\frac{h_n^{(1)\prime}(x)}{h_n^{(1)}(x)}\right\} < 0.$$

Proof. We can see form (55) and Lemma 4.9 that

$$\mathrm{Re}\left\{\frac{h_n^{(1)'}(x)}{h_n^{(1)}(x)}\right\} = -\frac{1}{2x} + \mathrm{Re}\left\{\frac{H_{n+1/2}^{(1)'}(x)}{H_{n+1/2}^{(1)}(x)}\right\} < 0. \quad \blacksquare$$

LEMMA **4.11** *For every $\nu \in \boldsymbol{R}$, $\left|H_\nu^{(1)}(x)\right|$ is a decreasing function on $(0, \infty)$.*

Proof. We can see from the proof of Lemma 4.9 that

$$\frac{d}{dx}\left|H_\nu^{(1)}(x)\right|^2 = 2\left(J_\nu(x)J_\nu'(x) + N_\nu(x)N_\nu'(x)\right) < 0.$$

This implies $\left|H_\nu^{(1)}(x)\right|$ is a decreasing function on $(0, \infty)$. $\blacksquare$

LEMMA **4.12** *For every $x \in (0, \infty)$ and for $\nu,\ \nu' \in \boldsymbol{R}$ satisfying $|\nu| < |\nu'|$, we have*

$$\left|H_\nu^{(1)}(x)\right| < \left|H_{\nu'}^{(1)}(x)\right|.$$

Proof. It is clear from (57) and (59). $\blacksquare$

LEMMA **4.13** *For every $n \in \boldsymbol{N} \cup \{0\}$, $\left|h_n^{(1)}(x)\right|$ is a decreasing function on $(0, \infty)$.*

Proof. It is clear from (19) and Lemma 4.11. $\blacksquare$

LEMMA **4.14** *For every $x \in (0, \infty)$ and for $n,\ n' \in \boldsymbol{N} \cup \{0\}$ satisfying $n < n'$, we have*

$$\left|h_n^{(1)}(x)\right| < \left|h_{n'}^{(1)}(x)\right|.$$

Proof. It is clear from (19) and Lemma 4.12. $\blacksquare$

5 Existence and uniqueness of the solution to the reduced problem

The weak problem of (2) is formulated as follows: find $u \in V$ such that

$$(61)\quad a(u,\, v) - k^2(u,\, v) + s(u,\, v) = (f,\, v) \qquad \text{for all } v \in V,$$

where

$$\begin{aligned}
a(u,\, v) &= \int_{\Omega_a} \nabla u \cdot \nabla \overline{v}\, dx \qquad \text{for } u,\ v \in H^1(\Omega_a),\\
(u,\, v) &= \int_{\Omega_a} u\overline{v}\, dx \qquad \text{for } u,\ v \in L^2(\Omega_a),\\
s(u,\, v) &= \langle \mathcal{S}u,\, v\rangle_{H^{-1/2}(\Gamma_a)\times H^{1/2}(\Gamma_a)} \qquad \text{for } u,\ v \in H^1(\Omega_a),\\
V &= \left\{v \in H^1(\Omega_a) \mid v = 0 \ \text{ on } \ \gamma\right\}.
\end{aligned}$$

THEOREM **5.1** *For every $f \in L^2(\Omega_a)$, problem (61) has a unique solution.*

Proof. We first prove in the two dimensional case. We define the sesquilinear form $b(\cdot,\,\cdot)$ as follows:

$$b(u,\,v) = \sum_{n=-\infty}^{\infty} |n| u_n(a)\overline{v_n(a)} \quad \text{for all } u, v \in V, \tag{62}$$

where $u_n(a)$ and $v_n(a)$ are the Fourier coefficients defined by (5) of u and v, respectively. We introduce a new inner product on V:

$$((u,\,v)) = a(u,\,v) + b(u,\,v),$$

and the associated norm:

$$|||u||| = ((u,\,u))^{1/2}.$$

Then we see that the norm $|||\cdot|||$ is equivalent to the standard $H^1(\Omega_a)$ norm. We define the bounded linear operator K_1 on V through

$$((K_1 u,\,v)) = -k^2(u,\,v) \quad \text{for all } u, v \in V.$$

By the compact imbedding of $H^1(\Omega_a)$ to $L^2(\Omega_a)$, we see K_1 is a compact operator on V.

We here note that $s(u,\,v)$ can be represented as follows:

$$s(u,\,v) = \sum_{n=-\infty}^{\infty} -ka \frac{H_n^{(1)\prime}(ka)}{H_n^{(1)}(ka)} u_n(a)\overline{v_n(a)}. \tag{63}$$

It follows from (62), (63), and Lemma 4.3 that there exists a positive constant C such that

$$|s(u,\,v) - b(u,\,v)| \leq C\|u\|_{L^2(\Gamma_a)}\|v\|_{L^2(\Gamma_a)} \quad \text{for all } u, v \in V. \tag{64}$$

Further the trace operator from $H^1(\Omega_a)$ into $L^2(\Gamma_a)$ is compact. Therefore we can define the compact operator K_2 on V through

$$((K_2 u,\,v)) = s(u,\,v) - b(u,\,v) \quad \text{for all } u, v \in V.$$

We can write the problem (61) as follows:

$$(I + K_1 + K_2)u = g, \tag{65}$$

where $g \in V$ such that

$$((g,\,v)) = (f,\,v) \quad \text{for all } v \in V.$$

Since $K_1 + K_2$ is a compact operator on V, in order to show the unique solvability of (65), it suffices to prove that $I + K_1 + K_2$ is one-to-one. Suppose that $u \in V$ satisfies

$$(I + K_1 + K_2)u = 0, \tag{66}$$

then we have

$$a(u,\,u) - k^2(u,\,u) + s(u,\,u) = 0.$$

Taking the imaginary part of this identity, we get

$$0 = \mathrm{Im}\{s(u,\,u)\} = \sum_{n=-\infty}^{\infty} -ka\,\mathrm{Im}\left\{\frac{H_n^{(1)\prime}(ka)}{H_n^{(1)}(ka)}\right\} |u_n(a)|^2.$$

It follows from Lemma 4.7 that $u_n(a) = 0$ for all $n \in \boldsymbol{Z}$. Hence $u \equiv 0$ on Γ_a, by the unique continuation property, $u \equiv 0$ on Ω_a. Therefore $I + K_1 + K_2$ is one-to-one.

We can analogously prove in the three dimensional case. In this case, we define $b(\cdot,\, \cdot)$ by

$$(67)\quad b(u,\, v) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a(n+1) u_n^m(a) \overline{v_n^m(a)} \qquad \text{for all } u, v \in V,$$

instead of (62). Since we have

$$(68)\quad s(u,\, v) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} -ka^2 \frac{h_n^{(1)\prime}(ka)}{h_n^{(1)}(ka)} u_n^m(a) \overline{v_n^m(a)},$$

from (67), (68), and Lemma 4.4, we can also get (64). Suppose that $u \in V$ satisfies (66), then

$$0 = \operatorname{Im}\{s(u,\, u)\} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} -ka^2 \operatorname{Im} \left\{ \frac{h_n^{(1)\prime}(ka)}{h_n^{(1)}(ka)} \right\} |u_n^m(a)|^2.$$

Thus, by Lemma 4.8, $u_n^m(a) = 0$ for all n and m. In the same argument as in the two dimensional case, we see $I + K_1 + K_2$ is one-to-one. ■

6 Equivalence between the original problem and its reduced problem

LEMMA **6.1** *Let $\varphi \in H^{1/2}(\Gamma_a)$. We define, for $r \geq a$,*

$$(69)\quad u = \begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \varphi_n Y_n(\theta) & \text{if} \quad d = 2, \\[2ex] \displaystyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \varphi_n^m Y_n^m(\theta,\, \phi) & \text{if} \quad d = 3, \end{cases}$$

where φ_n and φ_n^m are the Fourier coefficients of φ defined by (36) *and* (37), *respectively. Then $u \in H^1_{\mathrm{loc}}(\overline{\Omega'_a})$, and furthermore the series* (69) *and its term by term first derivatives converge in $L^2(\Omega_a^b)$ for every $b > a$. In addition, u satisfies*

$$(70)\quad \begin{cases} -\Delta u - k^2 u = f & \text{in } \Omega'_a, \\ u = \varphi & \text{on } \Gamma_a, \\ \displaystyle\lim_{r \longrightarrow +\infty} r^{\frac{d-1}{2}} \left(\frac{du}{dr} - iku \right) = 0. & \end{cases}$$

Proof. We first prove in the two dimensional case. For $N \in \boldsymbol{N}$, we set

$$u_N = \sum_{n=-N}^{N} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \varphi_n Y_n(\theta).$$

For $N < N'$, we have

$$\|u_N - u_{N'}\|^2_{L^2(\Omega_a^b)} = \sum_{N < |n| \leq N'} \int_a^b \left| \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \right|^2 r\, dr |\varphi_n|^2,$$

and further, by Lemma 4.11, we can get

$$\|u_N - u_{N'}\|^2_{L^2(\Omega_a^b)} \le \frac{1}{2}(b^2 - a^2) \sum_{N<|n|\le N'} |\varphi_n|^2.$$

Thus, since $\varphi \in H^{1/2}(\Gamma_a)$,

$$\|u_N - u_{N'}\|^2_{L^2(\Omega_a^b)} \longrightarrow 0 \quad (N,\, N' \longrightarrow \infty),$$

and hence

$$u_N \longrightarrow u \quad \text{ in } L^2(\Omega_a^b) \quad (N \longrightarrow \infty).$$

Now, we note

$$(71)\quad \int_{\Omega_a^b} \nabla u \cdot \nabla v \, dx = \int_{\Omega_a^b} \frac{\partial u}{\partial r}\frac{\partial v}{\partial r}\, dx - \int_{\Omega_a^b} \frac{1}{r^2}(\Lambda u) v \, dx$$

for $u,\, v \in C^\infty(\overline{\Omega_a^b})$, where Λ is the Laplace-Beltrami operator on the unit circle of $\boldsymbol{R}^2$. Thus, since $-\Lambda Y_n = n^2 Y_n$, we have

$$(72)\quad \|\nabla(u_N - u_{N'})\|^2_{L^2(\Omega_a^b)}$$
$$= \int_{\Omega_a^b} \left| \frac{\partial u_N}{\partial r} - \frac{\partial u_{N'}}{\partial r} \right|^2 dx - \int_{\Omega_a^b} \frac{1}{r^2} \left[\Lambda(u_N - u_{N'})\right] \left(\overline{u_N - u_{N'}}\right) dx$$
$$= \sum_{N<|n|\le N'} \int_a^b \left\{ \left| \frac{d}{dr} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \right|^2 + \frac{n^2}{r^2} \left| \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \right|^2 \right\} r\, dr |\varphi_n|^2.$$

We here have, by integration by parts,

$$\begin{aligned}(73)\int_a^b \left| \frac{d}{dr} H_n^{(1)}(kr) \right|^2 r\, dr &= \int_a^b \frac{d}{dr} H_n^{(1)}(kr) \frac{d}{dr} \overline{H_n^{(1)}(kr)} r \, dr \\ &= \left[H_n^{(1)}(kr) \frac{d}{dr} \overline{H_n^{(1)}(kr)} r \right]_a^b \\ &\quad - \int_a^b H_n^{(1)}(kr) \frac{d^2}{dr^2} \overline{H_n^{(1)}(kr)} r \, dr - \int_a^b H_n^{(1)}(kr) \frac{d}{dr} \overline{H_n^{(1)}(kr)} \, dr.\end{aligned}$$

Since $H_n^{(1)}$ satisfies the Bessel equation, we have

$$(74)\quad \frac{d^2}{dr^2} H_n^{(1)}(kr) = -\left\{ \frac{k}{r} H_n^{(1)\prime}(kr) + \left[k^2 - \frac{n^2}{r^2} \right] H_n^{(1)}(kr) \right\}.$$

Substituting (74) into (73), we can get

$$(75)\quad \int_a^b \left| \frac{d}{dr} H_n^{(1)}(kr) \right|^2 r\, dr$$
$$\begin{aligned} &= \left[H_n^{(1)}(kr) \frac{d}{dr} \overline{H_n^{(1)}(kr)} r \right]_a^b \\ &\quad + \int_a^b H_n^{(1)}(kr) \left\{ \frac{k}{r} \overline{H_n^{(1)\prime}(kr)} + \left[k^2 - \frac{n^2}{r^2} \right] \overline{H_n^{(1)}(kr)} \right\} r \, dr \\ &\quad - \int_a^b H_n^{(1)}(kr) \overline{H_n^{(1)\prime}(kr)} k \, dr \\ &= \left[H_n^{(1)}(kr) \overline{H_n^{(1)\prime}(kr)} kr \right]_a^b \\ &\quad + \int_a^b k^2 r \left| H_n^{(1)}(kr) \right|^2 dr - n^2 \int_a^b \frac{1}{r} |H_n^{(1)}(kr)|^2 \, dr.\end{aligned}$$

By (75), we get

$$\int_a^b \left\{ \left| \frac{d}{dr} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \right|^2 + \frac{n^2}{r^2} \left| \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \right|^2 \right\} r\, dr$$

$$= \left[\frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \overline{\left(\frac{H_n^{(1)'}(kr)}{H_n^{(1)}(ka)} \right)} kr \right]_a^b + \int_a^b k^2 r \left| \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \right|^2 dr \equiv I + II.$$

By using (47), we have

$$\begin{aligned} I &= \frac{H_n^{(1)}(kb)\overline{H_n^{(1)'}(kb)}}{\left|H_n^{(1)}(ka)\right|^2} kb - \overline{\left(\frac{H_n^{(1)'}(ka)}{H_n^{(1)}(ka)} \right)} ka \\ &= kb \frac{H_n^{(1)}(kb)}{\left|H_n^{(1)}(ka)\right|^2} \left\{ \overline{H_{n-1}^{(1)}(kb)} - \frac{n}{kb} \overline{H_n^{(1)}(kb)} \right\} - ka \frac{\overline{H_{n-1}^{(1)}(ka)} - \frac{n}{ka} \overline{H_n^{(1)}(ka)}}{\overline{H_n^{(1)}(ka)}} \\ &= kb \frac{H_n^{(1)}(kb)\overline{H_{n-1}^{(1)}(kb)}}{\left|H_n^{(1)}(ka)\right|^2} - n \left| \frac{H_n^{(1)}(kb)}{H_n^{(1)}(ka)} \right|^2 - ka \overline{\left(\frac{H_{n-1}^{(1)}(ka)}{H_n^{(1)}(ka)} \right)} + n. \end{aligned}$$

Thus, by assuming here $n > 0$ and by using Lemmas 4.11 and 4.12, we can get

$$\begin{aligned} |I| &\le n + k(a+b), \\ |II| &\le \frac{k^2}{2}(b^2 - a^2). \end{aligned}$$

From these estimates and the fact that $H_{-n}^{(1)}(x) = (-1)^n H_n^{(1)}(x)$ ($n \in \boldsymbol{Z}$), we can get, for all $n \in \boldsymbol{Z}$,

$$\begin{aligned} &\int_a^b \left\{ \left| \frac{d}{dr} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \right|^2 + \frac{n^2}{r^2} \left| \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \right|^2 \right\} r^2\, dr \\ &\le |n| + k(a+b) + \frac{k^2}{2}(b^2 - a^2). \end{aligned}$$

Thus, because of $\varphi \in H^{1/2}(\Gamma_a)$, the right-hand side of (72) tends to zero as $N,\, N' \longrightarrow \infty$. Therefore we can see $u_N \longrightarrow u$ in $H^1(\Omega_a^b)$. It is clear from the results above that u satisfies (70).

We can analogously prove in the three dimensional case. For $N \in \boldsymbol{N}$, we set

$$u_N = \sum_{n=0}^{N} \sum_{m=-n}^{n} \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \varphi_n^m Y_n^m(\theta,\, \phi).$$

For $N < N'$, we have

$$\|u_N - u_{N'}\|_{L^2(\Omega_a^b)}^2 = \sum_{n=N+1}^{N'} \sum_{m=-n}^{n} \int_a^b \left| \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \right|^2 r^2\, dr |\varphi_n^m|^2.$$

By Lemma 4.13, we can get

$$\|u_N - u_{N'}\|_{L^2(\Omega_a^b)}^2 \le \frac{1}{3}(b^3 - a^3) \sum_{n=N+1}^{N'} \sum_{m=-n}^{n} |\varphi_n^m|^2 \longrightarrow 0 \quad (N,\, N' \longrightarrow \infty).$$

This implies

$$u_N \longrightarrow u \quad \text{in } L^2(\Omega_a^b) \quad (N \longrightarrow \infty).$$

Now we note that (71) also holds in the three dimensional case. Thus, by using (71) and the fact that $-\Lambda Y_n^m = n(n+1)Y_n^m$, we can get

$$\begin{aligned}
(76)\quad & \|\nabla(u_N - u_{N'})\|^2_{L^2(\Omega_a^b)} \\
&= \int_{\Omega_a^b} \left| \frac{\partial u_N}{\partial r} - \frac{\partial u_{N'}}{\partial r} \right|^2 dx - \int_{\Omega_a^b} \frac{1}{r^2} \left[\Lambda(u_N - u_{N'})\right] (\overline{u_N - u_{N'}})\, dx \\
&= \sum_{n=N+1}^{N'} \sum_{m=-n}^{n} \int_a^b \left\{ \left| \frac{d}{dr} \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \right|^2 + \frac{n(n+1)}{r^2} \left| \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \right|^2 \right\} r^2\, dr |\varphi_n^m|^2.
\end{aligned}$$

We here have, by integration by parts,

$$\begin{aligned}
\int_a^b \left| \frac{d}{dr} h_n^{(1)}(kr) \right|^2 r^2\, dr &= \int_a^b \frac{d}{dr} h_n^{(1)}(kr) \frac{d}{dr} \overline{h_n^{(1)}(kr)} r^2\, dr \\
&= \left[h_n^{(1)}(kr) \frac{d}{dr} \overline{h_n^{(1)}(kr)} r^2 \right]_a^b \\
&\quad - \int_a^b h_n^{(1)}(kr) \frac{d^2}{dr^2} \overline{h_n^{(1)}(kr)} r^2\, dr - \int_a^b h_n^{(1)}(kr) \frac{d}{dr} \overline{h_n^{(1)}(kr)} 2r\, dr.
\end{aligned}$$

Since

$$(77)\quad \frac{d^2}{dr^2} h_n^{(1)}(kr) = -\left\{ \frac{2k}{r} h_n^{(1)\prime}(kr) + \left[k^2 - \frac{n(n+1)}{r^2} \right] h_n^{(1)}(kr) \right\},$$

we have

$$\begin{aligned}
(78)\quad & \int_a^b \left| \frac{d}{dr} h_n^{(1)}(kr) \right|^2 r^2\, dr \\
&= \left[h_n^{(1)}(kr) \frac{d}{dr} \overline{h_n^{(1)}(kr)} r^2 \right]_a^b \\
&\quad + \int_a^b h_n^{(1)}(kr) \left\{ \frac{2k}{r} \overline{h_n^{(1)\prime}(kr)} + \left[k^2 - \frac{n(n+1)}{r^2} \right] \overline{h_n^{(1)}(kr)} \right\} r^2\, dr \\
&\quad - \int_a^b h_n^{(1)}(kr) \overline{h_n^{(1)\prime}(kr)} 2kr\, dr \\
&= \left[h_n^{(1)}(kr) \overline{h_n^{(1)\prime}(kr)} kr^2 \right]_a^b \\
&\quad + \int_a^b (kr)^2 \left| h_n^{(1)}(kr) \right|^2 dr - n(n+1) \int_a^b |h_n^{(1)}(kr)|^2\, dr.
\end{aligned}$$

By (78), we get

$$\begin{aligned}
& \int_a^b \left\{ \left| \frac{d}{dr} \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \right|^2 + \frac{n(n+1)}{r^2} \left| \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \right|^2 \right\} r^2\, dr \\
&= \left[\frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \overline{\left(\frac{h_n^{(1)\prime}(kr)}{h_n^{(1)}(ka)} \right)} kr^2 \right]_a^b + \int_a^b (kr)^2 \left| \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \right|^2 dr \equiv I' + II'.
\end{aligned}$$

By using (51), we have

$$
\begin{aligned}
I' &= \frac{h_n^{(1)}(kb)\overline{h_n^{(1)'}(kb)}}{\left|h_n^{(1)}(ka)\right|^2} kb^2 - \overline{\left(\frac{h_n^{(1)'}(ka)}{h_n^{(1)}(ka)}\right)} ka^2 \\
&= kb^2 \frac{h_n^{(1)}(kb)}{\left|h_n^{(1)}(ka)\right|^2} \left\{ \overline{h_{n-1}^{(1)}(kb)} - \frac{n+1}{kb} \overline{h_n^{(1)}(kb)} \right\} - ka^2 \frac{\overline{h_{n-1}^{(1)}(ka)} - \frac{n+1}{ka} \overline{h_n^{(1)}(ka)}}{\overline{h_n^{(1)}(ka)}} \\
&= kb^2 \frac{h_n^{(1)}(kb) h_{n-1}^{(1)}(kb)}{\left|h_n^{(1)}(ka)\right|^2} - (n+1)b \left|\frac{h_n^{(1)}(kb)}{h_n^{(1)}(ka)}\right|^2 - ka^2 \overline{\left(\frac{h_{n-1}^{(1)}(ka)}{h_n^{(1)}(ka)}\right)} + a(n+1).
\end{aligned}
$$

Thus, by using Lemmas 4.13 and 4.14, we can get

$$
\begin{aligned}
|I'| &\leq (n+1)(a+b) + k(a^2+b^2), \\
|II'| &\leq \frac{k^2}{3}(b^3 - a^3).
\end{aligned}
$$

From these estimates, we can get

$$
\begin{aligned}
(79) \quad & \int_a^b \left\{ \left| \frac{d}{dr} \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \right|^2 + \frac{n(n+1)}{r^2} \left| \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \right|^2 \right\} r^2 \, dr \\
& \leq (n+1)(a+b) + k(a^2+b^2) + \frac{k^2}{3}(b^3 - a^3).
\end{aligned}
$$

From (76) , (79), and the condition that $\varphi \in H^{1/2}(\Gamma_a)$, we see that $\|\nabla(u_N - u_{N'})\|_{L^2(\Omega_a^b)} \longrightarrow 0$ as N, $N' \longrightarrow \infty$. Therefore we can see $u_N \longrightarrow u$ in $H^1(\Omega_a^b)$. ■

Lemma **6.2** *Let $\varphi \in H^{3/2}(\Gamma_a)$. We define u by (69). Then, for every $b > a$, we have*

$$
(80) \quad \frac{\partial u}{\partial r}(r, \theta) = \sum_{n=-\infty}^{\infty} k \frac{H_n^{(1)'}(kr)}{H_n^{(1)}(ka)} \varphi_n(a) Y_n(\theta) \quad \textit{in } H^1(\Omega_a^b) \quad (d = 2),
$$

$$
(81) \quad \frac{\partial u}{\partial r}(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} k \frac{h_n^{(1)'}(kr)}{h_n^{(1)}(ka)} \varphi_n^m(a) Y_n^m(\theta, \phi) \quad \textit{in } H^1(\Omega_a^b) \quad (d = 3).
$$

Proof. From Lemma 6.1 it follows from that the infinite series of the right-hand sides of (80) and (81) converge in $L^2(\Omega_a^b)$. Hence, it is sufficient to show that their first derivatives converge in $L^2(\Omega_a^b)$.

We first consider the two dimensional case. For every $N \in \boldsymbol{N}$, we define

$$
v_N = \sum_{n=-\infty}^{\infty} k \frac{H_n^{(1)'}(kr)}{H_n^{(1)}(ka)} \varphi_n(a) Y_n(\theta).
$$

For $N < N'$,

$$
\begin{aligned}
& \|\nabla(v_N - v_{N'})\|_{L^2(\Omega_a^b)}^2 \\
&= \sum_{N < |n| \leq N'} \int_a^b \left\{ \left| \frac{d^2}{dr^2} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \right|^2 + \frac{n^2}{r^2} \left| \frac{d}{dr} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \right|^2 \right\} r \, dr |\varphi_n|^2.
\end{aligned}
$$

By (74), we can get

$$\begin{aligned}\left|\frac{d^2}{dr^2}H_n^{(1)}(kr)\right|^2 &= \frac{1}{r^2}\left|\frac{d}{dr}H_n^{(1)}(kr)\right|^2 + \left[k^2 - \frac{n^2}{r^2}\right]^2 \left|H_n^{(1)}(kr)\right|^2 \\ &\quad +2\left[\frac{k^2}{r} - \frac{n^2}{r^3}\right]\mathrm{Re}\left\{\left(\frac{d}{dr}H_n^{(1)}(kr)\right)\overline{H_n^{(1)}(kr)}\right\}.\end{aligned}$$

Thus, by using (47) and Lemmas 4.11 and 4.12 in a similar way to the proof of Lemma 6.1, we have

$$\int_a^b \left|\frac{d^2}{dr^2}\frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)}\right|^2 r\,dr = n^4 \int_a^b \left|\frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)}\right|^2 \frac{dr}{r^3} + O(|n|^3) \quad (|n| \longrightarrow \infty). \tag{82}$$

We now have, by integration by parts,

$$\begin{aligned}&\int_a^b \left|\frac{d}{dr}H_n^{(1)}(kr)\right|^2 \frac{dr}{r} \\ &= \int_a^b \frac{d}{dr}H_n^{(1)}(kr)\frac{d}{dr}\overline{H_n^{(1)}(kr)}\,\frac{dr}{r} \\ &= \left[H_n^{(1)}(kr)\frac{d}{dr}\overline{H_n^{(1)}(kr)}\frac{1}{r}\right]_a^b \\ &\quad -\int_a^b H_n^{(1)}(kr)\frac{d^2}{dr^2}\overline{H_n^{(1)}(kr)}\,\frac{dr}{r} + \int_a^b H_n^{(1)}(kr)\frac{d}{dr}\overline{H_n^{(1)}(kr)}\,\frac{dr}{r^2}.\end{aligned}$$

By (74), we further have

$$\begin{aligned}&\int_a^b \left|\frac{d}{dr}H_n^{(1)}(kr)\right|^2 \frac{dr}{r} \\ &= \left[H_n^{(1)}(kr)\frac{d}{dr}\overline{H_n^{(1)}(kr)}\frac{1}{r}\right]_a^b \\ &\quad +\int_a^b H_n^{(1)}(kr)\frac{d}{dr}\overline{H_n^{(1)}(kr)}\,\frac{dr}{r^2} + \int_a^b \left|H_n^{(1)}(kr)\right|^2\left[k^2 - \frac{n^2}{r^2}\right]\frac{dr}{r} \\ &\quad +\int_a^b H_n^{(1)}(kr)\frac{d}{dr}\overline{H_n^{(1)}(kr)}\,\frac{dr}{r^2}.\end{aligned}$$

Thus, in the same way as above, we obtain

$$n^2\int_a^b \left|\frac{d}{dr}\frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)}\right|^2 \frac{dr}{r} = -n^4 \int_a^b \left|\frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)}\right|^2 \frac{dr}{r^3} + O(|n|^3) \quad (|n| \longrightarrow \infty). \tag{83}$$

From (82) and (83), we can get

$$\int_a^b \left\{\left|\frac{d^2}{dr^2}\frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)}\right|^2 + \frac{n^2}{r^2}\left|\frac{d}{dr}\frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)}\right|^2\right\} r\,dr = O(|n|^3) \quad (|n| \longrightarrow \infty).$$

Therefore, since $\varphi \in H^{3/2}(\Gamma_a)$, we can get (80).

We next consider the three dimensional case. For every $N \in \boldsymbol{N}$, we define

$$v_N = \sum_{n=0}^{N}\sum_{m=-n}^{n} k\frac{h_n^{(1)\prime}(kr)}{h_n^{(1)}(ka)}\varphi_n^m Y_n^m(\theta,\,\phi).$$

For $N < N'$,

$$\begin{aligned}
&\|\nabla(v_N - v_{N'})\|^2_{L^2(\Omega_a^b)} \\
&\quad = \sum_{n=N+1}^{N'} \sum_{m=-n}^{n} \int_a^b \left\{ \left| \frac{d^2}{dr^2} \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \right|^2 + \frac{n(n+1)}{r^2} \left| \frac{d}{dr} \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \right|^2 \right\} r^2\, dr |\varphi_n^m|^2.
\end{aligned}$$

By (77), we can get

$$\begin{aligned}
\left| \frac{d^2}{dr^2} h_n^{(1)}(kr) \right|^2 &= \frac{4}{r^2} \left| \frac{d}{dr} h_n^{(1)}(kr) \right|^2 + \left[k^2 - \frac{n(n+1)}{r^2} \right]^2 \left| h_n^{(1)}(kr) \right|^2 \\
&\quad + 4 \left[\frac{k^2}{r} - \frac{n(n+1)}{r^3} \right] \mathrm{Re} \left\{ \left(\frac{d}{dr} h_n^{(1)}(kr) \right) \overline{h_n^{(1)}(kr)} \right\}.
\end{aligned}$$

Thus, by using (51) and Lemmas 4.11 and 4.12 in a similar way to the proof of Lemma 6.1, we have

$$\int_a^b \left| \frac{d^2}{dr^2} \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \right|^2 r^2\, dr = n^4 \int_a^b \left| \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \right|^2 \frac{dr}{r^2} + O(n^3) \quad (n \longrightarrow \infty). \tag{84}$$

We now have, by integration by parts,

$$\begin{aligned}
&\int_a^b \left| \frac{d}{dr} h_n^{(1)}(kr) \right|^2 dr \\
&\quad = \int_a^b \frac{d}{dr} h_n^{(1)}(kr) \frac{d}{dr} \overline{h_n^{(1)}(kr)}\, dr \\
&\quad = \left[h_n^{(1)}(kr) \frac{d}{dr} \overline{h_n^{(1)}(kr)} \right]_a^b - \int_a^b h_n^{(1)}(kr) \frac{d^2}{dr^2} \overline{h_n^{(1)}(kr)}\, dr.
\end{aligned}$$

By (77), we further have

$$\begin{aligned}
&\int_a^b \left| \frac{d}{dr} h_n^{(1)}(kr) \right|^2 dr \\
&\quad = \left[h_n^{(1)}(kr) \frac{d}{dr} \overline{h_n^{(1)}(kr)} \right]_a^b \\
&\qquad + \int_a^b h_n^{(1)}(kr) \frac{d}{dr} \overline{h_n^{(1)}(kr)} \frac{2}{r}\, dr + \int_a^b \left| h_n^{(1)}(kr) \right|^2 \left[k^2 - \frac{n(n+1)}{r^2} \right] dr.
\end{aligned}$$

Thus, in the same way as above, we obtain

$$n(n+1) \int_a^b \left| \frac{d}{dr} \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \right|^2 dr = -n^4 \int_a^b \left| \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \right|^2 \frac{dr}{r^2} + O(n^3) \quad (n \longrightarrow \infty). \tag{85}$$

From (84) and (85), we can get

$$\int_a^b \left\{ \left| \frac{d^2}{dr^2} \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \right|^2 + \frac{n(n+1)}{r^2} \left| \frac{d}{dr} \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \right|^2 \right\} r^2\, dr = O(n^3) \quad (n \longrightarrow \infty).$$

Therefore, since $\varphi \in h^{3/2}(\Gamma_a)$, we can get (81). ■

THEOREM **6.1** *Let f be a function belonging to $L^2(\Omega)$ such that* $\operatorname{supp} f \subset \overline{\Omega_a}$*, and let u be the solution of* (1) *belonging to $H^1_{\rm loc}(\overline{\Omega})$. Then $u|_{\Omega_a}$ is the solution of* (61)*, and u is represented in Ω'_a as* (4) *and* (6) *in the two and three dimensional cases, respectively.*

Conversely, let u_i be the solution of (61) *and let $\varphi = u_i|_{\Gamma_a}$. We define $u \in L^2_{\rm loc}(\overline{\Omega})$ as follows:*

$$
\begin{aligned}
(86)\quad u|_{\Omega_a} &= u_i, \\
(87)\quad u|_{\Omega'_a} &= \begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \varphi_n Y_n(\theta) & \text{if} \quad d = 2, \\[2ex] \displaystyle\sum_{n=0}^{\infty} \sum_{|m|\le n} \frac{h_n^{(1)}(kr)}{h_n^{(1)}(ka)} \varphi_n^m Y_n^m(\theta,\, \phi) & \text{if} \quad d = 3, \end{cases}
\end{aligned}
$$

where φ_n and φ_n^m are the Fourier coefficients of φ defined by (36) *and* (37)*, respectively. Then $u \in H^1_{\rm loc}(\overline{\Omega})$, and u is the solution of* (1)*.*

Proof. We prove only in the two dimensional case, since in the three dimensional case we can also prove in exactly the same way. Let u be the solution of (1) belonging to $H^1_{\rm loc}(\overline{\Omega})$. By Lemma 2.2, we can see that u is represented in Ω'_a as (4) in the two dimensional case. Since, from the usual regularity argument, $u \in H^2_{\rm loc}(\overline{\Omega})$, we have $u|_{\Gamma_a} \in H^{3/2}(\Gamma_a)$. Hence, by Lemma 6.2, we have

$$
(88)\quad \frac{\partial u}{\partial r}(r,\, \theta) = \sum_{n=-\infty}^{\infty} k \frac{H_n^{(1)\prime}(kr)}{H_n^{(1)}(ka)} u_n(a) Y_n(\theta) \quad \text{in } H^1(\Omega_a^b)
$$

for every $b > a$. Since $-\Delta u - k^2 u = f$ in Ω_a, by using the Green formula, we have

$$
(89)\quad \int_{\Omega_a} (\nabla u \cdot \nabla \overline{v} - k^2 u \overline{v})\, dx - \int_{\Gamma_a} \frac{\partial u}{\partial n} \overline{v}\, d\gamma = \int_{\Omega_a} f \overline{v}\, dx \quad \text{for all } v \in V.
$$

From (88) we can see

$$
(90)\quad -\int_{\Gamma_a} \frac{\partial u}{\partial n} \overline{v}\, d\gamma = \sum_{n=-\infty}^{\infty} -k \frac{H_n^{(1)\prime}(ka)}{H_n^{(1)}(ka)} u_n(a) \int_0^{2\pi} \overline{v} Y_n(\theta) a\, d\theta = s(u,\, v).
$$

Combining (89) and (90), we can see $u|_{\Omega_a}$ is the solution of (61).

Now, let u_i be the solution of (61), and we define $u \in L^2_{\rm loc}(\overline{\Omega})$ by (86) and (87). Then we will show

$$
(91)\quad \int_{\Omega} u(-\Delta \overline{\psi} - k^2 \overline{\psi})\, dx = \int_{\Omega} f \overline{\psi}\, dx \quad \text{for all } \psi \in C_0^\infty(\Omega).
$$

We denote $u|_{\Omega'_a}$ by u_e. By the Green formula, we have

$$
\begin{aligned}
\int_{\Omega} u(-\Delta \overline{\psi} - k^2 \overline{\psi})\, dx &= \int_{\Omega_a} u_i(-\Delta \overline{\psi} - k^2 \overline{\psi})\, dx + \int_{\Omega'_a} u_e(-\Delta \overline{\psi} - k^2 \overline{\psi})\, dx \\
&= -\int_{\Gamma_a} u_i \frac{\partial \overline{\psi}}{\partial r}\, d\gamma + \int_{\Omega_a} (\nabla u_i \cdot \nabla \overline{\psi} - k^2 u_i \overline{\psi})\, dx \\
&\quad + \int_{\Gamma_a} u_e \frac{\partial \overline{\psi}}{\partial r}\, d\gamma + \int_{\Omega'_a} (\nabla u_e \cdot \nabla \overline{\psi} - k^2 u_e \overline{\psi})\, dx.
\end{aligned}
$$

Here noting $u_i = u_e$ on Γ_a, we obtain

$$(92)\quad \int_\Omega u(-\Delta\overline{\psi} - k^2\overline{\psi})\,dx = \int_{\Omega_a} f\overline{\psi}\,dx - s(u_i,\,\psi) + \int_{\Omega'_a}(\nabla u_e\cdot\nabla\overline{\psi} - k^2 u_e\overline{\psi})\,dx.$$

We here have

$$(93)\quad \int_{\Omega'_a}(\nabla u_e\cdot\nabla\overline{\psi} - k^2 u_e\overline{\psi})\,dx = s(u_i,\,\psi).$$

Indeed, for $N \in \boldsymbol{N}$, we define

$$u_N = \sum_{n=-N}^{N} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)}\varphi_n Y_n(\theta),$$

where φ_n is the Fourier coefficients of $\varphi \equiv u_i|_{\Gamma_a}$. Then, $u_N \in C^\infty(\overline{\Omega'_a})$ and we have

$$-\Delta u_N - k^2 u_N = 0 \quad \text{ in } \Omega'_a,$$

by the Green formula, we obtain

$$(94)\quad \int_{\Omega'_a}(\nabla u_N\cdot\nabla\overline{\psi} - k^2 u_N\overline{\psi})\,dx + \sum_{n=-N}^{N} k\frac{H_n^{(1)'}(ka)}{H_n^{(1)}(ka)}\varphi_n \int_{\Gamma_a} Y_n(\theta)\overline{\psi}\,d\gamma = 0.$$

There exists a $b > a$ such that

$$\operatorname{supp}\psi \cap \Omega'_a \subset \Omega_a^b.$$

Then we can rewrite (94) as follows:

$$(95)\quad \int_{\Omega_a^b}(\nabla u_N\cdot\nabla\overline{\psi} - k^2 u_N\overline{\psi})\,dx + \sum_{n=-N}^{N} k\frac{H_n^{(1)'}(ka)}{H_n^{(1)}(ka)}\varphi_n \int_{\Gamma_a} Y_n(\theta)\overline{\psi}\,d\gamma = 0.$$

Since $u_i|_{\Gamma_a} \in H^{1/2}(\Gamma_a)$, we obtain

$$\sum_{n=-\infty}^{\infty} ka\frac{H_n^{(1)'}(ka)}{H_n^{(1)}(ka)}\varphi_n\overline{\psi_n} = -s(u_i,\,\psi),$$

and by Lemma 6.1, we can see

$$u_N \longrightarrow u_e \quad \text{ in } H^1(\Omega_a^b).$$

Thus, by letting N tend to infinity in (95), we can see (93) holds. From (92) and (93), we can see (91) holds. Further it is clear that $u = 0$ on γ and that u satisfies the outgoing radiation condition. Therefore we conclude that u is the solution of (1). ■

We now have the following corollary to Theorem 6.1:

Corollary **6.1** *For every $f \in L^2(\Omega_a)$, the unique solution u of problem* (61) *belongs to $H^2(\Omega_a)$. Further we have the following a priori estimate:*

$$(96)\quad \|u\|_{H^2(\Omega_a)} \le C_r\|f\|_{L^2(\Omega_a)},$$

where C_r is a positive constant independent of f and u.

Proof. We see from Theorem 5.1 that problem (61) has a unique solution $u \in V$. By virtue of Theorem 6.1, u can be extend to $\Omega \setminus \Omega_a$ so as to be the solution to problem (1) which belongs to $H^2_{\mathrm{loc}}(\overline{\Omega})$. Thus we can conclude $u \in H^2(\Omega_a)$. We now define the operator $G : L^2(\Omega_a) \longrightarrow H^2(\Omega_a)$ as follows: for every $f \in L^2(\Omega_a)$,

$$Gf = u,$$

where u is the solution to problem (61) with f. Since we can readily show G to be a closed operator, we can get, by the closed graph theorem, the a priori estimate (96). ■

Remark **6.1** We consider the exterior problem with the *incoming* radiation condition:

$$\left\{\begin{array}{rcll} -\Delta u - k^2 u &=& f & \text{in } \Omega, \\ u &=& 0 & \text{on } \gamma, \\ \lim\limits_{r \longrightarrow +\infty} r^{\frac{d-1}{2}} \left(\dfrac{du}{dr} + iku \right) &=& 0. & \end{array}\right.$$

For this problem, all the results described above hold with appropriate modifications. We then note that the analytical representations, in Ω'_a, of the incoming solutions are:

$$u = \left\{\begin{array}{ll} \displaystyle\sum_{n=-\infty}^{\infty} \frac{H_n^{(2)}(kr)}{H_n^{(2)}(ka)} \varphi_n Y_n(\theta) & \text{if} \quad d = 2, \\ \displaystyle\sum_{n=0}^{\infty} \sum_{|m| \le n} \frac{h_n^{(2)}(kr)}{h_n^{(2)}(ka)} \varphi_n^m Y_n^m(\theta,\, \phi) & \text{if} \quad d = 3, \end{array}\right.$$

and the DtN operators corresponding to the incoming radiation condition are:

$$\mathcal{S}^* \varphi = \left\{\begin{array}{ll} \displaystyle\sum_{n=-\infty}^{\infty} -k \frac{H_n^{(2)\prime}(ka)}{H_n^{(2)}(ka)} \varphi_n Y_n & \text{if} \quad d = 2, \\ \displaystyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} -k \frac{h_n^{(2)\prime}(ka)}{h_n^{(2)}(ka)} \varphi_n^m Y_n^m & \text{if} \quad d = 3. \end{array}\right. \tag{97}$$

In addition, the fundamental solutions satisfying the incoming radiation condition are:

$$\psi(x) = \left\{\begin{array}{ll} -\dfrac{i}{4} H_0^{(2)}(k|x|) & \text{if} \quad d = 2, \\ \dfrac{1}{4\pi} \dfrac{e^{-ik|x|}}{|x|} & \text{if} \quad d = 3. \end{array}\right.$$

7 Finite element approximation

We can get numerical solutions to problem (1) by applying the finite element method to problem (61). Such a method is called the *DtN method* and is studied by several authors (Goldstein [4], Masmoudi [9], Keller and Givoli [7], Harari and Hughes [6], Grote and Keller [5], and Bao [2], for instance). We establish well-posedness of the discrete problem obtained by applying the finite element method to problem (61) and error estimates for solutions to the discrete problem. To do so, we follow the idea due to Goldstein [4], which is also used in [2].

In this section we denote the norm and the semi-norm of $H^m(\Omega_a)$ ($m \in \boldsymbol{N} \cup \{0\}$) with $H^0(\Omega_a) = L^2(\Omega_a)$ by $\|\cdot\|_{m,\Omega_a}$ and $|\cdot|_{m,\Omega_a}$, respectively, and further we define the norm of $H^s(\Gamma_a)$ ($s > 0$) as follows: for $\varphi \in H^s(\Gamma_a)$,

$$\|\varphi\|^2_{s,\Gamma_a} = \begin{cases} \displaystyle\sum_{n=-N}^{N} (1 + |n|^{2s})|\varphi_n|^2 & \text{if} \quad d = 2, \\ \displaystyle\sum_{n=0}^{N} \sum_{m=-n}^{n} (1 + |n|^{2s})|\varphi_n^m|^2 & \text{if} \quad d = 3. \end{cases}$$

We consider a family $\{V_h \,|\, h \in (0,\, \bar{h}]\}$ of finite dimensional subspaces of V such that for all $0 < h \leq \bar{h}$ and for every $u \in V \cap H^2(\Omega_a)$,

$$\text{(98)} \quad \inf_{v^h \in V_h} |u - v_h|_{1,\Omega_a} \leq C_a h \|u\|_{2,\Omega_a},$$

where C_a is a positive constant independent of h and u. If $d = 2$, such a family $\{V_h \,|\, h \in (0,\, \bar{h}]\}$ can be constructed by using the curved elements due to Zlámal [13]. (Since Γ_a is a circle, we need to consider the curved elements.) We briefly explain how to construct such a family. For each $h \in (0,\, \bar{h}]$, we consider a triangulation $\mathcal{T}_h$ of Ω_a including curved elements near the curved parts of $\partial\Omega_a$. Let V_h be a conforming finite element space associated with $\mathcal{T}_h$. Here every function of V_h is supposed to be a linear function on each interior triangle element and to be a certain function introduced by Zlámal on each curved element. Suppose that the family $\{\mathcal{T}_h \,|\, h \in (0,\, \bar{h}]\}$ is regular in the sense of Ciarlet [3]. Then the family $\{V_h \,|\, h \in (0,\, \bar{h}]\}$ satisfies (98).

Then the discrete problem of problem (61) associated with V_h is as follows: find $u_h \in V_h$ such that

$$\text{(99)} \quad a(u_h,\, v_h) - k^2(u_h,\, v_h) + s(u_h,\, v_h) = (f,\, v_h) \qquad \text{for all } v_h \in V_h.$$

Well-posedness of problem (99) and error estimates for $u - u_h$ for sufficiently small h are established in the following theorem:

THEOREM **7.1** *Let k be an arbitrary positive number, f an arbitrary function of $L^2(\Omega_a)$, and u the solution to problem* (61)*. Then there exists an $h_0(k) \in (0,\, \bar{h}]$ such that for every $0 < h \leq h_0(k)$, problem* (99) *has a unique solution u_h, and further*

$$\text{(100)} \quad |u - u_h|_{1,\Omega_a} \leq C_1(k) h \|f\|_{0,\Omega_a},$$
$$\text{(101)} \quad \|u - u_h\|_{0,\Omega_a} \leq C_2(k) h^2 \|f\|_{0,\Omega_a},$$

where $h_0(k)$, $C_1(k)$, and $C_2(k)$ are independent of f, u, and u_h, and further $C_1(k)$ and $C_2(k)$ are independent of h.

Proof. We first assume that problem (99) has a solution u_h. A proof of the well-posedness of problem (99) is postpone to the completion of the derivation of the error estimates (100) and (101).

Set $e_h = u - u_h$. Then we have

$$\text{(102)} \quad \tilde{a}(e_h,\, v_h) = 0$$

for all $v_h \in V_h$, where

$$\text{(103)} \quad \tilde{a}(u,\, v) = a(u,\, v) - k^2(u,\, v) + s(u,\, v)$$

for $u,\, v \in H^1(\Omega_a)$. Note the following identical equation:

$$|e_h|_{1,\Omega_a}^2 = k^2\|e_h\|_{0,\Omega_a}^2 - s(e_h,\, e_h) + \tilde{a}(e_h,\, e_h).$$

Taking the real part of this identity, we can get

$$|e_h|_{1,\Omega_a}^2 = k^2\|e_h\|_{0,\Omega_a}^2 - \operatorname{Re} s(e_h,\, e_h) + \operatorname{Re} \tilde{a}(e_h,\, e_h).$$

By virtue of Lemmas 4.9 and 4.10, we get

$$(104)\ |e_h|_{1,\Omega_a}^2 \le k^2\|e_h\|_{0,\Omega_a}^2 + \operatorname{Re} \tilde{a}(e_h,\, e_h).$$

To estimate the right-hand side of (104), we use the Poincaré inequality

$$(105)\ \|v\|_{0,\Omega_a} \le C_p|v|_{1,\Omega_a} \quad \text{for all } v \in V$$

and the trace inequality

$$(106)\ \|v\|_{1/2,\Gamma_a} \le C_t|v|_{1,\Omega_a} \quad \text{for all } v \in V.$$

Step 1. In this step, we show that there exists a positive constant $C_3(k)$ such that

$$(107)\ |\tilde{a}(e_h,\, e_h)| \le C_3(k)h|e_h|_{1,\Omega_a}\|u\|_{2,\Omega_a},$$

where $C_3(k)$ is independent of h, u, and u_h. We see from (102) and (103) that for all $v_h \in V_h$,

$$\tilde{a}(e_h,\, e_h) = a(e_h,\, u - v_h) - k^2(e_h,\, u - v_h) + s(e_h,\, u - v_h).$$

By the trigonometric inequality, the Schwarz inequality, and the boundedness of $\mathcal{S} : H^{1/2}(\Gamma_a) \longrightarrow H^{-1/2}(\Gamma_a)$,

$$\begin{aligned} &|\tilde{a}(e_h,\, e_h)| \\ &\quad \le \ |e_h|_{1,\Omega_a}|u - v_h|_{1,\Omega_a} + k^2\|e_h\|_{0,\Omega_a}\|u - v_h\|_{0,\Omega_a} + \|\mathcal{S}\|\|e_h\|_{1/2,\Gamma_a}\|u - v_h\|_{1/2,\Gamma_a}. \end{aligned}$$

By (105) and (106),

$$|\tilde{a}(e_h,\, e_h)| \le C_3(k)|e_h|_{1,\Omega_a}|u - v_h|_{1,\Omega_a},$$

where

$$(108)\ C_3(k) = C_p^2k^2 + \|\mathcal{S}\|C_t^2 + 1.$$

From this inequality and (98), we can get (107).

Step 2. In this step, we show that there exists a positive constant $C_4(k)$ such that

$$(109)\ \|e_h\|_{0,\Omega_a} \le C_4(k)h|e_h|_{1,\Omega_a},$$

where $C_4(k)$ is independent of h, u, and u_h. Suppose that $w \in V$ satisfies

$$(110)\ a(w,\, v) - k^2(w,\, v) + \overline{s(v,\, w)} = (e_h,\, v)$$

for all $v \in V$. Then w is a weak solution of the following problem:

$$\left\{ \begin{array}{rcll} -\Delta u - k^2 u &=& e_h & \text{in } \Omega_a, \\ u &=& 0 & \text{on } \gamma, \\ \dfrac{\partial u}{\partial n} &=& -\mathcal{S}^* u & \text{on } \Gamma_a, \end{array} \right.$$

where $\mathcal{S}^*$ is the DtN operator corresponding to the incoming radiation condition given by (97). As mentioned in Remark 6.1, Corollary 6.1 holds for the incoming solution w. Hence we have $w \in H^2(\Omega_a)$ and

(111) $\|w\|_{2,\Omega_a} \leq C_r^* \|e_h\|_{0,\Omega_a}$.

Taking $v = e_h$ in (110), we obtain

$$\begin{aligned} \|e_h\|_{0,\Omega_a}^2 &= a(w,\, e_h) - k^2(w,\, e_h) + \overline{s(e_h,\, w)} \\ &= \overline{a(e_h,\, w) - k^2(e_h,\, w) + s(e_h,\, w)}, \end{aligned}$$

and hence, by (102), we have for all $v_h \in V_h$,

$$\|e_h\|_{0,\Omega_a}^2 = \overline{a(e_h,\, w - v_h) - k^2(e_h,\, w - v_h) + s(e_h,\, w - v_h)}.$$

In the same way as in Step 1, we can get

$$\begin{aligned} &\|e_h\|_{0,\Omega_a}^2 \\ &\quad \leq |e_h|_{1,\Omega_a} |w - v_h|_{1,\Omega_a} + k^2 \|e_h\|_{0,\Omega_a} \|w - v_h\|_{0,\Omega_a} + \|\mathcal{S}\| \|e_h\|_{1/2,\Gamma_a} \|w - v_h\|_{1/2,\Gamma_a} \\ &\quad \leq C_3(k) |e_h|_{1,\Omega_a} |w - v_h|_{1,\Omega_a}, \end{aligned}$$

where $C_3(k)$ is the constant given by (108). From this inequality, (98), and (111), we can get (109) with $C_4(k) = C_3(k) C_r^* C_a$.

Combining (104), (107), and (109), we obtain

$$|e_h|_{1,\Omega_a} \leq C_5(k) h^2 |e_h|_{1,\Omega_a} + C_3(k) h \|u\|_{2,\Omega_a},$$

where $C_5(k) = k^2 (C_4(k))^2$. Thus, for every $h \in (0,\, \bar{h}]$ satisfying

$$1 - C_5(k) h^2 \geq \frac{1}{2},$$

which is equivalent to

$$0 < h \leq \frac{1}{\sqrt{2 C_5(k)}},$$

we have

(112) $|e_h|_{1,\Omega_a} \leq \dfrac{C_3(k)}{2} h \|u\|_{2,\Omega_a}$.

Let here $h_0(k) = \min(1/\sqrt{2C_5(k)},\, \bar{h})$. We can see from (112) and (96) that for every $0 < h \leq h_0(k)$, we have (100) with $C_1(k) = C_3(k) C_r / 2$. Further, combining (109) and (100), we obtain (101) with $C_2(k) = C_1(k) C_4(k)$.

We next show the well-posedness of problem (99). For this purpose, it is sufficient to show uniqueness of the solution to problem (99) since V_h is finite dimensional. Hence assume that $u_h \in V_h$ is a solution to problem (99) with $f = 0$. Since the solution u to problem (61) with $f = 0$ is identically zero, it follows from (100) (or (101)) that $u_h = 0$. Thus we can conclude that problem (99) is well-posed. ■

Remark **7.1** We can see from the proof of Theorem 7.1 that $h_0(k)$ is a decreasing function of k on $(0,\, \infty)$ and $C_1(k)$ and $C_2(k)$ are increasing functions of k on $(0,\, \infty)$.

7.1 Truncation of the DtN operator

Practically we can not compute the problem (99) because the sesquilinear form s is analytically represented by the infinite series. Hence, in the practical computations of numerical solutions, we have to truncate this infinite series. We analyze the effect of this truncation on the error estimates. To this end, we introduce the following sesquilinear form: for $N \in \boldsymbol{N}$,

$$s^N(u,\, v) = \begin{cases} \displaystyle\sum_{n=-N}^{N} -ka\frac{H_n^{(1)\prime}(ka)}{H_n^{(1)}(ka)} u_n(a)\overline{v_n(a)} & \text{if} \quad d=2, \\ \displaystyle\sum_{n=0}^{N}\sum_{m=-n}^{n} -ka^2\frac{h_n^{(1)\prime}(ka)}{h_n^{(1)}(ka)} u_n^m(a)\overline{v_n^m(a)} & \text{if} \quad d=3. \end{cases}$$

We here consider the following problem: find $u_h^N \in V_h$ such that

(113) $a(u_h^N,\, v_h) - k^2(u_h^N,\, v_h) + s^N(u_h^N,\, v_h) = (f,\, v_h) \quad$ for all $v_h \in V_h$.

We show that this problem is well-posed for h sufficiently small and for N sufficiently large, and further we establish error estimates for $u - u_h^N$ when h is sufficiently small and N is sufficiently large. That is, we prove the following theorem:

THEOREM **7.2** *Let k be an arbitrary positive number, f an arbitrary function of $L^2(\Omega_a)$, and u the solution to problem (61). Then there exist a $\gamma_0(k) > 0$ such that for every $(h,\, N) \in (0,\, \bar{h}] \times \boldsymbol{N}$ satisfying $h + N^{-1} \le \gamma_0(k)$, problem (113) has a unique solution u_h^N, and further*

(114) $|u - u_h^N|_{1,\Omega_a} \le C_1(k)(h + N^{-1})\|f\|_{0,\Omega_a}$,

(115) $\|u - u_h^N\|_{0,\Omega_a} \le C_2(k)(h + N^{-1})^2\|f\|_{0,\Omega_a}$,

where $\gamma_0(k)$, $C_1(k)$, and $C_2(k)$ are independent of f, u, and u_h^N, and further $C_1(k)$ and $C_2(k)$ are independent of h and N.

To prove Theorem 7.2, we here introduce two sesquilinear forms on $H^1(\Omega_a)$:

$$\begin{aligned} \tilde{a}^N(u,\, v) &= a(u,\, v) - k^2(u,\, v) + s^N(u,\, v), \\ r^N(u,\, v) &= \begin{cases} \displaystyle\sum_{|n|>N} -ka\frac{H_n^{(1)\prime}(ka)}{H_n^{(1)}(ka)} u_n(a)\overline{v_n(a)} & \text{if} \quad d=2, \\ \displaystyle\sum_{n>N}\sum_{m=-n}^{n} -ka^2\frac{h_n^{(1)\prime}(ka)}{h_n^{(1)}(ka)} u_n^m(a)\overline{v_n^m(a)} & \text{if} \quad d=3. \end{cases} \end{aligned}$$

Note here that we have

$$s(u,\, v) = s^N(u,\, v) + r^N(u,\, v) \quad \text{for } u,\, v \in H^1(\Omega_a).$$

Proof. As in the proof of Theorem 7.1, we first assume that problem (113) has a solution u_h^N.

Set $e_h^N = u - u_h^N$. Then we have

(116) $\tilde{a}^N(e_h^N,\, v_h) + r^N(u,\, v_h) = 0$

for all $v_h \in V_h$. Note the following identical equation:

$$|e_h^N|_{1,\Omega_a}^2 = k^2\|e_h^N\|_{0,\Omega_a}^2 - s^N(e_h^N,\, e_h^N) + \tilde{a}^N(e_h^N,\, e_h^N).$$

Taking the real part of this identity, we can get

$$|e_h^N|_{1,\Omega_a}^2 = k^2\|e_h^N\|_{0,\Omega_a}^2 - \operatorname{Re} s^N(e_h^N,\, e_h^N) + \operatorname{Re} \tilde{a}^N(e_h^N,\, e_h^N).$$

By virtue of Lemmas 4.9 and 4.10, we get

$$(117)\ |e_h^N|_{1,\Omega_a}^2 \le k^2\|e_h^N\|_{0,\Omega_a}^2 + \operatorname{Re} \tilde{a}^N(e_h^N,\, e_h^N).$$

We here introduce the set of all polynomials of degree l in the variable k with non-negative coefficients:

$$P_l^+ = \left\{ \sum_{j=0}^{l} a_j k^j \mid a_l > 0,\ a_j \ge 0\, (0 \le j \le l-1) \right\}.$$

Step 1. In this step, we show that there exist positive constants $C_3(k)$, C_4, and C_5 such that for an arbitrary $\varepsilon > 0$,

$$(118)\ |\tilde{a}^N(e_h^N,\, e_h^N)| \le \varepsilon |e_h^N|_{1,\Omega_a}^2 + \left(\frac{C_3(k)}{\varepsilon} h^2 + \frac{C_4}{\varepsilon} N^{-2} + C_5 h N^{-1} \right) \|u\|_{2,\Omega_a}^2,$$

where constants $C_3(k)$, C_4, and C_5 are independent of h, N, u, u_h^N, and ε, and further C_4 and C_5 are independent of k. Here $C_3(k)$ belongs to P_4^+ as a function of k. By (116), we have for all $v_h \in V_h$,

$$\begin{aligned} \tilde{a}^N(e_h^N,\, e_h^N) &= \tilde{a}^N(e_h^N,\, u - v_h) + r^N(u,\, u_h^N - v_h) \\ &= \tilde{a}^N(e_h^N,\, u - v_h) + r^N(u,\, u - v_h) - r^N(u,\, e_h^N). \end{aligned}$$

Thus, by using the trigonometric inequality, the Schwarz inequality, and Lemmas 4.5 and 4.6, we get

$$\begin{aligned} &|\tilde{a}^N(e_h^N,\, e_h^N)| \\ &\le |e_h^N|_{1,\Omega_a}|u - v_h|_{1,\Omega_a} + k^2\|e_h^N\|_{0,\Omega_a}\|u - v_h\|_{0,\Omega_a} + C\|e_h^N\|_{1/2,\Gamma_a}\|u - v_h\|_{1/2,\Gamma_a} \\ &\quad + |r^N(u,\, u - v_h)| + |r^N(u,\, e_h^N)|. \end{aligned}$$

Further, by (105) and (106),

$$(119)\ |\tilde{a}^N(e_h^N,\, e_h^N)| \le C_6(k)|e_h^N|_{1,\Omega_a}|u - v_h|_{1,\Omega_a} + |r^N(u,\, u - v_h)| + |r^N(u,\, e_h^N)|,$$

where $C_6(k) \in P_2^+$. Let us here estimate the second term on the right-hand side of (119). If $d = 2$, by Lemma 4.5, the Schwarz inequality, and the trace theorem,

$$\begin{aligned} (120)\ |r^N(u,\, u - v_h)| &\le C \sum_{|n|>N} |n||u_n(a)||(u - v_h)_n(a)| \\ &\le CN^{-1}\|u\|_{3/2,\Gamma_a}\|u - v_h\|_{1/2,\Gamma_a} \\ &\le CN^{-1}\|u\|_{2,\Omega_a}\|u - v_h\|_{1,\Omega_a}, \end{aligned}$$

where $(u - v_h)_n(a)$ are the Fourier coefficients of $u - v_h$. If $d = 3$, we analogously get (120) by using Lemma 4.6 instead of Lemma 4.5. Further, for the third term, we can similarly estimate as follows:

$$(121)\ |r^N(u,\, e_h^N)| \le CN^{-1}\|u\|_{2,\Omega_a}\|e_h^N\|_{1,\Omega_a}.$$

Combining (119), (120), (121), and (98), we achieve the following:

$$\begin{aligned}&|\tilde{a}^N(e_h^N,\, e_h^N)|\\ &\quad\le\; C_7(k)h|e_h^N|_{1,\Omega_a}\|u\|_{2,\Omega_a} + C_8N^{-1}|e_h^N|_{1,\Omega_a}\|u\|_{2,\Omega_a} + C_9hN^{-1}\|u\|^2_{2,\Omega_a},\end{aligned}$$

where $C_7(k) \in P_2^+$. Applying the arithmetic-geometric mean inequality to the first and second terms on the right-hand side of the above inequality, we have, for an arbitrary $\varepsilon > 0$,

$$\begin{aligned}|\tilde{a}^N(e_h^N,\, e_h^N)| \;&\le\; \frac{\varepsilon}{2}|e_h^N|^2_{1,\Omega_a} + \frac{C_{10}(k)}{\varepsilon}h^2\|u\|^2_{2,\Omega_a}\\ &\quad+\frac{\varepsilon}{2}|e_h^N|^2_{1,\Omega_a} + \frac{C_{11}}{\varepsilon}N^{-2}\|u\|^2_{2,\Omega_a} + C_9hN^{-1}\|u\|^2_{2,\Omega_a},\end{aligned}$$

where $C_{10}(k) \in P_4^+$. This implies (118).

Step 2. In this step, we show that there exist positive constants $C_{12}(k)$ and C_{13} such that

$$(122)\; \|e_h^N\|_{0,\Omega_a} \le C_{12}(k)(h + N^{-1})|e_h^N|_{1,\Omega_a} + C_{13}(hN^{-1} + N^{-2})\|u\|_{2,\Omega_a},$$

where $C_{12}(k)$ and C_{13} are independent of h, N, u, and u_h^N, and further C_{13} is independent of k. Here $C_{12}(k)$ belongs to P_2^+ as a function of k. Suppose that $w \in V$ satisfies

$$(123)\; a(w,\, v) - k^2(w,\, v) + \overline{s(v,\, w)} = (e_h^N,\, v)$$

for all $v \in V$. Then w is the incoming solution. As mentioned in the proof of Theorem 7.1, we have $w \in H^2(\Omega_a)$ and

$$(124)\; \|w\|_{2,\Omega_a} \le C\|e_h^N\|_{0,\Omega_a}.$$

Taking $v = e_h^N$ in (123), we obtain

$$(125)\; \|e_h^N\|^2_{0,\Omega_a} = a(w,\, e_h^N) - k^2(w,\, e_h^N) + \overline{s(e_h^N,\, w)}.$$

Note here that (116) can be rewritten as follows:

$$(126)\; 0 = a(v_h,\, e_h^N) - k^2(v_h,\, e_h^N) + \overline{s^N(e_h^N,\, v_h)} + \overline{r^N(u,\, v_h)}.$$

Subtracting (126) from (125) gives

$$\begin{aligned}\|e_h^N\|^2_{0,\Omega_a} \;&=\; a(w - v_h,\, e_h^N) - k^2(w - v_h,\, e_h^N) + \overline{s^N(e_h^N,\, w - v_h)}\\ &\quad+\overline{r^N(e_h^N,\, w)} - \overline{r^N(u,\, v_h)}\\ &=\; a(w - v_h,\, e_h^N) - k^2(w - v_h,\, e_h^N) + \overline{s^N(e_h^N,\, w - v_h)}\\ &\quad+\overline{r^N(e_h^N,\, w)} + \overline{r^N(u,\, w - v_h)} - \overline{r^N(u,\, w)}.\end{aligned}$$

By the trigonometric inequality, the Schwarz inequality, and Lemmas 4.5 and 4.6,

$$\begin{aligned}&\|e_h^N\|^2_{0,\Omega_a}\\ &\quad=\; |w - v_h|_{1,\Omega_a}|e_h^N|_{1,\Omega_a} + k^2\|w - v_h\|_{0,\Omega_a}\|e_h^N\|_{0,\Omega_a} + C\|w - v_h\|_{1/2,\Gamma_a}\|e_h^N\|_{1/2,\Gamma_a}\\ &\qquad+|r^N(e_h^N,\, w)| + |r^N(u,\, w - v_h)| + |r^N(u,\, w)|.\end{aligned}$$

By (105) and (106),

$$(127)\ \|e_h^N\|^2_{0,\Omega_a} \leq C_{14}(k)|w-v_h|_{1,\Omega_a}|e_h^N|_{1,\Omega_a} + |r^N(e_h^N, w)| + |r^N(u, w-v_h)| + |r^N(u, w)|,$$

where $C_{14}(k) \in P_2^+$. We can here estimate the last three terms on th right-hand side of (127) as follows:

$$(128)\ |r^N(e_h^N, w)| \leq CN^{-1}\|e_h^N\|_{1/2,\Gamma_a}\|w\|_{3/2,\Gamma_a} \leq CN^{-1}|e_h^N|_{1,\Omega_a}\|w\|_{2,\Omega_a},$$

$$(129)\ |r^N(u, w-v_h)| \leq CN^{-1}\|u\|_{3/2,\Gamma_a}\|w-v_h\|_{1/2,\Gamma_a} \leq CN^{-1}\|u\|_{2,\Omega_a}|w-v_h|_{1,\Omega_a},$$

$$(130)\ |r^N(u, w)| \leq CN^{-2}\|u\|_{3/2,\Gamma_a}\|w\|_{3/2,\Gamma_a} \leq CN^{-2}\|u\|_{2,\Omega_a}\|w\|_{2,\Omega_a}.$$

Combining (127)–(130), we get

$$\|e_h^N\|^2_{0,\Omega_a} \leq \left\{C_{14}(k)|e_h^N|_{1,\Omega_a} + CN^{-1}\|u\|_{2,\Omega_a}\right\}|w-v_h|_{1,\Omega_a} + \left\{CN^{-1}|e_h^N|_{1,\Omega_a} + CN^{-2}\|u\|_{2,\Omega_a}\right\}\|w\|_{2,\Omega_a}.$$

Using here (98) and (124), we get

$$\|e_h^N\|^2_{0,\Omega_a} \leq \left\{C_{15}(k)|e_h^N|_{1,\Omega_a} + CN^{-1}\|u\|_{2,\Omega_a}\right\}h\|e_h^N\|_{0,\Omega_a} + \left\{CN^{-1}|e_h^N|_{1,\Omega_a} + CN^{-2}\|u\|_{2,\Omega_a}\right\}\|e_h^N\|_{0,\Omega_a},$$

where $C_{15}(k) \in P_2^+$, and further dividing by $\|e_h^N\|_{0,\Omega_a}$, we arrive at (122).

From (122), we can readily deduce

$$(131)\ \|e_h^N\|^2_{0,\Omega_a} \leq C_{16}(k)(h+N^{-1})^2|e_h^N|^2_{1,\Omega_a} + C(hN^{-1}+N^{-2})^2\|u\|^2_{2,\Omega_a},$$

where $C_{16}(k) \in P_4^+$. Combining (117), (118), and (131), we get

$$\left\{1-\varepsilon-C_{17}(k)(h+N^{-1})^2\right\}|e_h^N|^2_{1,\Omega_a} \leq C_{18}(k,\varepsilon)(h+N^{-1})^2\|u\|^2_{2,\Omega_a},$$

and further, by taking $\varepsilon = 1/2$,

$$\left\{\frac{1}{2}-C_{17}(k)(h+N^{-1})^2\right\}|e_h^N|^2_{1,\Omega_a} \leq C_{19}(k)(h+N^{-1})^2\|u\|^2_{2,\Omega_a},$$

where $C_{17}(k) \in P_6^+$ and $C_{18}(k, \varepsilon)$ and $C_{19}(k) \in P_4^+$. For every $\{h, N\} \in (0, \bar{h}] \times \boldsymbol{N}$ satisfying

$$\frac{1}{2} - C_{17}(k)(h+N^{-1})^2 \geq \frac{1}{4},$$

which is equivalent to

$$h + N^{-1} \leq \frac{1}{\sqrt{4C_{17}(k)}} \equiv \gamma_0(k),$$

we have

(132) $|e_h^N|_{1,\Omega_a} \leq C_{20}(k)(h+N^{-1})\|u\|_{2,\Omega_a}$,

where $C_{20}(k) \in P_4^+$. From (132) and (96), we get (114). Further, combining (122), (114), and (96), we obtain (115).

We can now deduce from (114) (or (115)) the well-posedness of problem (113) in the same argument as in the proof of Theorem 7.1. ■

Remark **7.2** We can see from the proof of Theorem 7.2 that $\gamma_0(k)$ is a decreasing function of k on $(0, \infty)$ and $C_1(k)$ and $C_2(k)$ are increasing functions of k on $(0, \infty)$.

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