Application of a Fictitious Domain Method to 3D Helmholtz Problems

Daisuke KOYAMA
Department of Computer Science
The University of Electro-Communications
小 山 大 介
電気通信大学 情報工学科

1 Introduction

We consider to compute numerical solutions of the three-dimensional exterior Helmholtz problem:

$$(1)\quad \begin{cases} -\Delta u - k^2 u = 0 & \text{in } \boldsymbol{R}^3 \setminus \overline{\mathcal{O}}, \\ u = g & \text{on } \gamma, \\ \displaystyle\lim_{r \longrightarrow +\infty} r \left(\frac{\partial u}{\partial r} - iku \right) = 0 & \text{(Sommerfeld radiation condition)}, \end{cases}$$

where k is a positive constant called the wave number, $\mathcal{O}$ is a bounded domain of $\boldsymbol{R}^3$ with Lipschitz continuous boundary γ, $\boldsymbol{R}^3 \setminus \mathcal{O}$ is assumed to be connected, $r = |x|$ ($x \in \boldsymbol{R}^3$), and $i = \sqrt{-1}$. This problem arises in models of acoustic scattering by a sound-soft obstacle $\mathcal{O}$ embedded in a homogeneous medium.

To compute numerical solutions of (1), we use a fictitious domain method with a Lagrange multiplier defined on γ, which is studied in [5], [6], [7], [8]. So we introduce a rectangular parallelepiped domain Ω, *the fictitious domain*, such that $\overline{\mathcal{O}} \subset \Omega$, and then we set $\omega = \Omega \setminus \overline{\mathcal{O}}$ and $\Gamma = \partial\Omega$ (see Figure 1). To approximate the Sommerfeld radiation condition in (1), we impose the Sommerfeld-like boundary condition on Γ:

$$\frac{\partial u}{\partial n} - iku = 0,$$

where n is the outward unit normal vector to Γ. This boundary condition is not so accurate; however, we do not discuss more accurate boundary condition here, for which we refer the reader to [1], [10]. As an approximate problem to (1), we here consider the following problem:

$$(2)\quad \begin{cases} -\Delta u - k^2 u = 0 & \text{in } \omega, \\ u = g & \text{on } \gamma, \\ \dfrac{\partial u}{\partial n} - iku = 0 & \text{on } \Gamma. \end{cases}$$

We can equivalently rewrite (2) as a saddle point problem in Ω which is obtained by extending the solution u of (2) to Ω so that the extended function also satisfies the homogeneous Helmholtz equation in $\mathcal{O}$, and by imposing weakly the non-homogeneous Dirichlet boundary condition on γ with a Lagrange multiplier. When we discretize such a saddle point problem, we may use a uniform tetrahedral mesh in Ω; however, we need to construct a triangular mesh on γ. These meshes can be constructed independently of each other, except that the boundary mesh size is larger than the mesh size in the domain. Thus the mesh generation in the fictitious domain method is easier than that in the usual finite element computations, especially when ω is a complicated shape. When the P_1 conforming finite element on Ω and the P_0 finite element on γ are used, *the constrain matrix* of the discrete saddle point problem, i.e., the matrix whose entries are integrals of the product of basis functions of the P_1 and P_0 finite elements, can be automatically computed with an algorithm introduced in Section 5. Furthermore, the use of uniform meshes in Ω allows us to use fast Helmholtz solvers as introduced in [3].

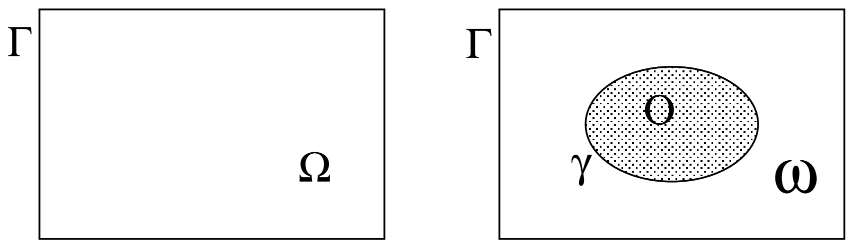

Figure 1: Domains Ω and ω etc.

We present an a priori error estimate for approximate solutions obtained by the fictitious domain method. Such an a priori error estimate is derived by following an idea of Girault and Glowinski [5]. Although they studied a positive definite Helmholtz problem, we here study an indefinite one. Thus our proof for the error estimate is slightly different from theirs; however, we do not write it here, which will be described in a forthcoming article. We further present results of numerical experiments concerning the rate of convergence for approximate solutions of a test problem which confirm the obtained a priori error estimate.

Girault et al. [6] analyze the error of the fictitious domain method applied to a nonhomogeneous steady incompressible Navier-Stokes problem. Bespalov [2], Kuznetsov-Lipnikov [11], Heikkola et al. [9], [10] study another fictitious domain method, which requires locally fitted meshes. Farhat et al. [4] propose a fictitious domain decomposition method aimed at solving efficiently partially axisymmetric acoustic scattering problems.

The remainder of this article is organized as follows. In Section 2, we describe the fictitious domain formulation of problem (2) and present a theorem concerning the well-posedness of the resulting saddle point problem. In Section 3, we formulate a discrete problem of the saddle point problem. In Section 4, we present the a priori error estimate mentioned above which are derived under some assumptions with respect to meshes in Ω and on γ and the regularity for the solution of the continuous saddle point problem. In Section 5, we describe how to compute the constrain matrix. In Section 6, we report results of numerical experiments, which are consistent with the a priori error estimate.

2 Fictitious domain formulation

A weak formulation of (2) is:

$$(3) \quad \begin{cases} \text{Find } u \in H^1(\omega) \text{ such that} \\ a(u, v) = 0 \quad \text{for all } v \in V, \\ \qquad u = g \quad \text{on } \gamma, \end{cases}$$

where $V = \{v \in H^1(\omega) \mid v = 0 \ \text{ on } \gamma\}$ and

$$a(u, v) = \int_\omega \left(\nabla u \cdot \nabla \overline{v} - k^2 u\overline{v}\right) dx - ik \int_\Gamma u\overline{v}\, d\gamma.$$

THEOREM **1** *For every $g \in H^{1/2}(\gamma)$, problem* (3) *has a unique solution.*

We here introduce some notations. We denote the standard Sobolev space $H^1(\Omega)$ by X. Let $H^{-1/2}(\gamma)$ be the set of all semi-linear forms on $H^{1/2}(\gamma)$. We denote $H^{-1/2}(\gamma)$ by M, and the duality pairing between $H^{-1/2}(\gamma)$ and $H^{1/2}(\gamma)$ by $\langle\cdot, \cdot\rangle_\gamma$.

The solution of (3) can be obtained by solving the following saddle point problem:

$$(4) \quad \begin{cases} \text{Find } \{u, \lambda\} \in X \times M \text{ such that} \\ \tilde{a}(u, v) + \overline{b(v, \lambda)} = 0 \qquad \text{for all } v \in X, \\ \overline{b(u, \mu)} = \langle \mu, g\rangle_\gamma \quad \text{for all } \mu \in M, \end{cases}$$

where

$$\tilde{a}(u, v) = \int_\Omega \left(\nabla u \cdot \nabla \overline{v} - k^2 u\overline{v}\right) dx - ik \int_\Gamma u\overline{v}\, d\gamma \qquad \text{for } u, v \in X,$$

$$b(v, \mu) = \overline{\langle \mu, v\rangle_\gamma} \qquad \text{for } v \in X \text{ and for } \mu \in M.$$

To describe the well-posedness of problem (4), we consider the following eigenvalue problem:

$$(5) \quad \begin{cases} -\Delta u = \alpha u \quad \text{in } \mathcal{O}, \\ \qquad u = 0 \quad \text{on } \gamma. \end{cases}$$

We denote by σ the set of all eigenvalues of (5).

THEOREM **2** *Assume that $k^2 \in (0, \infty) \setminus \sigma$. Then, for every $g \in H^{1/2}(\gamma)$, problem* (4) *has a unique solution $\{u, \lambda\} \in H^1(\Omega) \times H^{-1/2}(\gamma)$. Further the restriction of u to ω is the solution of problem* (3).

3 Discrete problem

We divide Ω by a uniform cube grid and subdivide each cube into six tetrahedrons, as in Figure 2. Let h denote the length of the longest edge of these tetrahedrons and let $\mathcal{T}_h$ denote the corresponding *tetrahedrization* of Ω. We take a Cartesian coordinate system in $\boldsymbol{R}^3$ so that Ω can be represented as follows: $\Omega = (-l_x/2,\, l_x/2) \times (-l_y/2,\, l_y/2) \times (-l_z/2,\, l_z/2)$. Let

$$\mathcal{H} = \left\{ h = \sqrt{3}h' \mid h' = \frac{l_x}{N_x} = \frac{l_y}{N_y} = \frac{l_z}{N_z},\ (N_x,\, N_y,\, N_z) \in \boldsymbol{N}^3 \right\}.$$

We consider a family $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$ of such tetrahedrizations of Ω. For each $h \in \mathcal{H}$, we take

$$X_h = \left\{ v_h \in C^0(\overline{\Omega}) \mid v_h|_T \in P_1 \text{ for every } T \in \mathcal{T}_h \right\},$$

where P_1 denotes the space of polynomials, in three variables, of degree less than or equal to one.

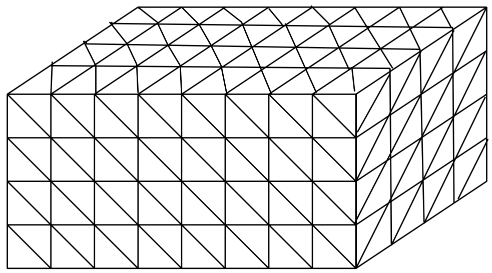

Figure 2: Tetrahedrization of domain Ω.

We here assume

(B) the boundary γ is polyhedral, with restrictions that its angles at edges and vertices are not too small.

We divide each face of γ into triangular patches. Let η be the maximum length of the sides of these triangular patches and denote by $\mathcal{P}_\eta$ the corresponding triangulation of γ. We consider a family $\{\mathcal{P}_\eta\}_{0<\eta\leq\bar{\eta}}$ of triangulations of γ. For each $\eta \in (0,\, \bar{\eta}]$, we take

$$M_\eta = \{\mu_\eta \mid \mu_\eta|_P \text{ is a constant for every } P \in \mathcal{P}_\eta\}.$$

A discrete problem of (4) is:

$$(6) \quad \left\{ \begin{array}{lcll} \multicolumn{4}{l}{\text{Find } \{u_h,\, \lambda_\eta\} \in X_h \times M_\eta \text{ such that}} \\ \tilde{a}(u_h,\, v_h) + \overline{b(v_h,\, \lambda_\eta)} & = & 0 & \text{for all } v_h \in X_h, \\ \overline{b(u_h,\, \mu_\eta)} & = & \langle \mu_\eta,\, g\rangle_\gamma & \text{for all } \mu_\eta \in M_\eta. \end{array} \right.$$

4 Error estimate

We assume the following:

$(H1)$ There exists a positive constant θ_0 independent of $\eta \in (0,\, \bar{\eta}]$ such that $\theta_P \geq \theta_0$ for all $P \in \mathcal{P}_\eta$, where θ_P is the smallest angle of P.

$(H2)$ There exists a positive constant L such that $\eta \leq Lh$.

$(H3)$ For every $P \in \mathcal{P}_\eta$, the diameter of the inscribed circle of P is grater than $4h$.

For the solution $\{u, \lambda\} \in X \times M$ of (4), we assume

$(R1)$ There exists an $s \in (1/2, 1]$ such that $u \in H^{1+s}(\Omega)$;

$(R2)$ $\lambda \in L^2(\gamma)$.

We now consider the following auxiliary problem: for a given $f \in L^2(\Omega)$, find $\{u, \lambda\} \in H^1(\Omega) \times H^{-1/2}(\gamma)$ such that

$$(7) \quad \left\{ \begin{array}{lcll} \tilde{a}^*(u, v) + \overline{b(v, \lambda)} & = & (f, v)_{L^2(\Omega)} & \text{for all } v \in X, \\ b(u, \mu) & = & 0 & \text{for all } \mu \in M, \end{array} \right.$$

where

$$\tilde{a}^*(u, v) = \int_\Omega \left(\nabla u \cdot \nabla \overline{v} - k^2 u \overline{v} \right) \, dx + ik \int_\Gamma u \overline{v} \, d\gamma.$$

For every $f \in L^2(\Omega)$, problem (7) has a unique solution. We assume that for every $f \in L^2(\Omega)$, the solution $\{u, \lambda\} \in X \times M$ of (7) satisfies

$(R3)$ $u \in H^{1+s}(\Omega)$, where s is the constant presented in $(R1)$;

$(R4)$ $\lambda \in L^2(\gamma)$.

Theorem **3** *Assume that hypotheses (B) and $(H1)$–$(H3)$ hold. Suppose that the wave number k satisfies $k^2 \in (0, \infty) \setminus \sigma$ and that hypotheses $(R1)$–$(R4)$ hold. Then, there exist positive constants $\bar{h}(k)$ and $\bar{\eta}(k)$ such that for all $\{h, \eta\} \in (0, \bar{h}(k)) \times (0, \bar{\eta}(k))$, problem* (6) *has a unique solution $\{u_h, \lambda_\eta\} \in X_h \times M_\eta$, and there exists a positive constant C such that*

$$(8) \quad \|u - u_h\|_{H^1(\Omega)} + \|\lambda - \lambda_\eta\|_{H^{-1/2}(\gamma)} \leq C \left\{ h^s \|u\|_{H^{1+s}(\Omega)} + \sqrt{\eta} \|\lambda\|_{L^2(\gamma)} \right\}.$$

5 Numerical computation

Let $\varphi_1, \ldots, \varphi_\mathcal{N}$ be the basis functions of X_h such that $\varphi_n(Q_l) = \delta_{nl}$ $(1 \leq n, l \leq \mathcal{N})$, where $\mathcal{N} = \dim X_h$, Q_l $(1 \leq l \leq \mathcal{N})$ are the nodes of tetrahedrization $\mathcal{T}_h$, and δ_{nl} denotes Kronecker's delta. Also let $\psi_1, \ldots, \psi_M$ be the basis functions of M_η such that $\psi_m|_{P_l} \equiv \delta_{ml}$ $(1 \leq m, l \leq \mathcal{M})$, where $\mathcal{M} = \dim M_\eta$ and P_l $(1 \leq l \leq \mathcal{M})$ are the triangular patches of triangulation $\mathcal{P}_\eta$. Then the solution $\{u_h, \lambda_\eta\}$ of problem (6) is written as follows:

$$u_h = \sum_{n=1}^{\mathcal{N}} c_n \varphi_n \quad \text{and} \quad \lambda_\eta = \sum_{m=1}^{\mathcal{M}} d_m \psi_m$$

with $(c_n)_{1 \leq n \leq \mathcal{N}} \in \boldsymbol{C}^\mathcal{N}$ and $(d_m)_{1 \leq m \leq \mathcal{M}} \in \boldsymbol{C}^\mathcal{M}$, and problem (6) is reduced to the following linear system:

$$\begin{bmatrix} A & B^T \\ B & O \end{bmatrix} \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} \boldsymbol{o} \\ \boldsymbol{g} \end{bmatrix},$$

where

$$A = (\tilde{a}(\varphi_n,\, \varphi_l))_{1 \le l,\, n \le \mathcal{N}}\,, \quad B = (b(\varphi_n,\, \psi_m))_{\,1 \le m \le \mathcal{M}, 1 \le n \le \mathcal{N}}\,,$$

$$\boldsymbol{c} = (c_n)_{1 \le n \le \mathcal{N}}, \quad \boldsymbol{d} = (d_m)_{1 \le m \le \mathcal{M}},$$

$$\boldsymbol{g} = \left(\overline{\langle \psi_m,\, g\rangle_\gamma}\right)_{1 \le m \le \mathcal{M}}.$$

Computation of matrix A is easy because uniform meshes are used in Ω; however, computation of matrix B is not so easy at first glance, so we will explain how to compute matrix B in the subsequent subsection.

5.1 Computation of matrix B

We first note that the $(n,\, m)$-entries of matrix B are given by

$$b(\varphi_n,\, \psi_m) = \int_{P_m} \varphi_n \, d\gamma.$$

To compute these values exactly, we need to construct a triangulation of the intersection of triangular patch P_m and each of tetrahedral elements of which the support of φ_n consists. We give an algorithm for constructing such a triangulation. We fix a triangular patch P and a tetrahedral element K, which are considered to be closed sets.

Algorithm for constructing a triangulation of $P \cap K$:

1. Compute the plane Π which includes the triangular patch P.
2. Seek $\Pi \cap K$ whose measure is positive.
   - 2-1. Count the number N_0 of vertices of K which are on Π and the number N_+ of vertices of K which are above Π. The cases for $(N_0,\, N_+)$ are listed in Table 1.
   - 2-2. Compute the intersection points of Π and edges of K which are not vertices of K. Their number N_i is written in Table 1.
   - 2-3. If $\Pi \cap K$ is a triangle, then proceed to the next procedure.
     If $\Pi \cap K$ is a quadrangle, then divide it into two triangles and proceed to the next procedure.
     If the measure of $\Pi \cap K$ is zero, then the measure of $P \cap K$ is also zero, and hence need not construct a triangulation of $P \cap K$.

   Thus, if the measure of $\Pi \cap K$ is positive then we can obtain one or two triangles, which will be denoted by T in the following, and are also considered to be closed.

Table 1: $\Pi \cap K$ and the number N_i of the intersection points of Π and edges of K which are not vertices of K are listed for each $(N_0,\ N_+)$, where N_0 is the number of the vertices of K which are on Π, and N_+ is the number of the vertices of K which are above Π.

N_0	N_+	$\Pi \cap K$	N_i
0	0	empty	0
	1	triangle	3
	2	quadrangle	4
	3	triangle	3
	4	empty	0
1	0	point	0
	1	triangle	2
	2	triangle	2
	3	point	0
2	0	line segment	0
	1	triangle	1
	2	line segment	0
3	0	triangle	0
	1	triangle	0

3. Construct a triangulation of $P \cap T$.

   Let s_1, s_2, s_3 be the sides of the triangle T, and let l_j $(j = 1,\ 2,\ 3)$ be the line including s_j. Let D_j be the closed half-plane on Π divided by l_j which includes the vertex of T not on l_j (see Figure 3). We here note that we have

$$T \cap P = \left(\bigcap_{j=1}^{3} D_j \right) \cap P = D_3 \cap (D_2 \cap (D_1 \cap P)).$$

   From this relation, we get the following procedure for constructing a triangulation of $T \cap P$.

   3-1. Construct a triangulation of $D_1 \cap P$.

   (a) Seek the line l_1.

   (b) Count the number n_0 of vertices of P which are on l_1 and the number n_+ of vertices of P which are interior points of D_1. There are cases for $(n_0,\ n_+)$ as in Table 2.

   (c) Compute the intersection points of l_1 and sides of P which are not vertices of P. Their number n_i is written in Table 2.

   (d) If $D_1 \cap P$ is a triangle, which will be denoted by P_1, then proceed to procedure 3-2.
   If $D_1 \cap P$ is a quadrangle, then divide it into two triangles $P_1^{(1)}$ and $P_1^{(2)}$, and proceed to procedure 3-2.
   If the measure of $D_1 \cap P$ is zero, then the measure of $T \cap P$ is also zero, and hence need not construct a triangulation of $T \cap P$.

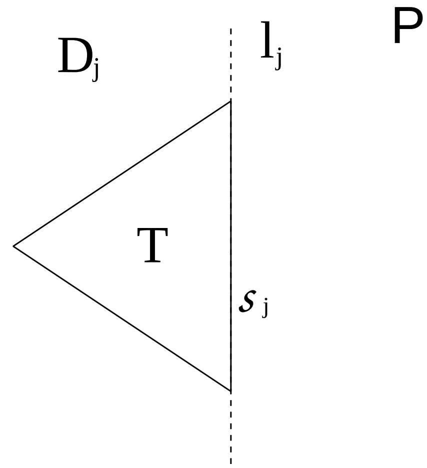

Figure 3: Half-plane D_j, triangle T, side s_j and line l_j.

Table 2: $D_1 \cap P$ and the number n_i of the intersection points of l_1 and sides of P which are not vertices of P are listed for each $(n_0,\, n_+)$, where n_0 is the number of the vertices of P which are on l_1, and n_+ is the number of the vertices of P which are interior points of D_1.

n_0	n_+	$D_1 \cap P$	n_i
0	3	triangle	0
	2	quadrangle	2
	1	triangle	2
	0	empty	0
1	2	triangle	0
	1	triangle	1
	0	point	0
2	1	triangle	0
	0	line segment	0

3-2. Construct a triangulation of $D_2 \cap (D_1 \cap P)$.
If $D_1 \cap P$ is a triangle, then we have

$$D_2 \cap (D_1 \cap P) = D_2 \cap P_1,$$

and hence apply procedure 3-1 to $D_2 \cap P_1$.
If $D_1 \cap P$ is a quadrangle, then we have

$$D_2 \cap (D_1 \cap P) = (D_2 \cap P_1^{(1)}) \cup (D_2 \cap P_1^{(2)}),$$

and hence apply procedure 3-1 to $D_2 \cap P_1^{(1)}$ and $D_2 \cap P_1^{(2)}$.

3-3. Construct a triangulation of $D_3 \cap (D_2 \cap (D_1 \cap P)) = T \cap P$ in the same way as in procedure 3-2.

Implementing this algorithm in a computer, we can automatically construct a triangulation of $K \cap P$.

6 Numerical experiments

We measure convergence rates of approximate solutions for a test problem whose exact solution is known analytically. In the problem, the boundary γ is a regular octahedron with length of the edges equal to 1.5, $\Omega = (-2, 2)^3$, and the wave number $k = 0.4$. The test problem is:

$$\begin{cases} \text{Find } \{u, \lambda\} \in X \times M \text{ such that} & & & \\ \tilde{a}(u, v) + \overline{b(v, \lambda)} & = & \displaystyle\int_\Omega F\overline{v}\, dx + \int_\Gamma f\overline{v}\, d\gamma & \text{for all } v \in X, \\ \overline{b(u, \mu)} & = & \langle \mu, g\rangle_\gamma & \text{for all } \mu \in M, \end{cases}$$

where the data F, f and g are so chosen that the exact solution becomes

$$u(x, y, z) = x^2 + y^2 + z^2 + i(x^2 - y^2 - z^2) \quad \text{in } \Omega,$$

which belongs to $C^\infty(\overline{\Omega})$, and then the Lagrange multiplier $\lambda = 0$ since λ is given by

$$\lambda = \frac{\partial u|_\omega}{\partial \nu} - \frac{\partial u|_{\mathcal{O}}}{\partial \nu},$$

where ν is the unit normal vector to γ outward from $\mathcal{O}$. This problem is associated with the following problem:

$$\begin{cases} -\Delta u - k^2 u & = & F & \text{in } \omega, \\ u & = & g & \text{on } \gamma, \\ \dfrac{\partial u}{\partial n} - iku & = & f & \text{on } \Gamma. \end{cases}$$

Although we have considered the case where $F = f = 0$ in the above sections, all the theorems stated above hold for the case where F and f are non-homogeneous, with proper modifications.

In our numerical experiments, mesh sizes h and η satisfy $h, \eta \leq (2\pi/k)/10$, i.e., the used meshes include at least ten grid points per the wavelength, which is a commonly used criterion for computing appropriate numerical solutions of the Helmholtz problem. In addition, the diameter of inscribed circle of each triangular patch is taken to be equal to $4h$ in order that hypothesis $(H3)$ is satisfied. All computations were performed in double precision complex arithmetic on VT-Alpha6 G IV personal computer (Alpha21264 800MHz CPU, 4GB Memory).

We report errors measured with $H^1(\Omega)$-seminorm and $L^2(\Omega)$-norm in Table 3, which shows that the rates of convergence with respect to $H^1(\Omega)$-seminorm and $L^2(\Omega)$-norm are $O(h^1)$ and $O(h^2)$, respectively. This convergence rate with respect to $H^1(\Omega)$-seminorm is consistent with error estimate (8) since $u \in H^2(\Omega)$ and $\lambda = 0$ in this test problem.

References

[1] Bamberger, Alain; Joly, Patrick; Roberts, Jean E.: Second-order absorbing boundary conditions for the wave equation: a solution for the corner problem. SIAM J. Numer. Anal. 27 (1990), no. 2, 323–352.

Table 3: Errors with respect to $H^1(\Omega)$-seminorm and $L^2(\Omega)$-norm.

h	$\dim X_h$	η	$\dim M_\eta$	$\lvert u-u_h\rvert_{H^1(\Omega)}$	$\lVert u-u_h\rVert_{L^2(\Omega)}$
$\sqrt{3}/8$	$3,5937$	$3/2$	8	1.42	5.02×10^{-2}
$\sqrt{3}/16$	$274,625$	$3/4$	32	0.70	1.26×10^{-2}
$\sqrt{3}/32$	$2,146,689$	$3/8$	128	0.35	3.16×10^{-3}

[2] Bespalov, A. N.: Application of algebraic fictitious domain method to the solution of 3D electromagnetic scattering problems. Russian J. Numer. Anal. Math. Modelling 12 (1997), no. 3, 211–229.

[3] Elman, Howard C; O'Leary, Dianne P.: Efficient iterative solution of the three-dimensional Helmholtz equation. J. Comput. Phys. 142 (1998), 163–181.

[4] Farhat, Charbel; Hetmaniuk, Ulrich: A fictitious domain decomposition method for the solution of partially axisymmetric acoustic scattering problems. I. Dirichlet boundary conditions. Internat. J. Numer. Methods Engrg. 54 (2002), no. 9, 1309–1332.

[5] Girault, V.; Glowinski, R.: Error analysis of a fictitious domain method applied to a Dirichlet problem. Japan J. Indust. Appl. Math. 12 (1995), no. 3, 487–514.

[6] Girault, V.; Glowinski, R.; López, H.; Vila, J.-P.: A boundary multiplier/fictitious domain method for the steady incompressible Navier-Stokes equations. Numer. Math. 88 (2001), no. 1, 75–103.

[7] Glowinski, Roland; Pan, Tsorng-Whay; Périaux, Jacques: A fictitious domain method for Dirichlet problem and applications. Comput. Methods Appl. Mech. Engrg. 111 (1994), no. 3-4, 283–303.

[8] Glowinski, Roland; Pan, Tsorng-Whay; Périaux, Jacques: A fictitious domain method for external incompressible viscous flow modeled by Navier-Stokes equations. Finite element methods in large-scale computational fluid dynamics (Minneapolis, MN, 1992). Comput. Methods Appl. Mech. Engrg. 112 (1994), no. 1-4, 133–148.

[9] Heikkola, Erkki; Kuznetsov, Yuri A.; Neittaanmäki, Pekka; Toivanen, Jari: Fictitious domain methods for the numerical solution of two-dimensional scattering problems. J. Comput. Phys. 145 (1998), no. 1, 89–109.

[10] Heikkola, Erkki; Kuznetsov, Yuri A.; Lipnikov, Konstantin N.: Fictitious domain methods for the numerical solution of three-dimensional acoustic scattering problems. J. Comput. Acoust. 7 (1999), no. 3, 161–183.

[11] Kuznetsov, Yu. A.; Lipnikov, K. N.: 3D Helmholtz wave equation by fictitious domain method. Russian J. Numer. Anal. Math. Modelling 13 (1998), no. 5, 371–387.