Report CS 00–01

Study on the Wave Equation with an Artificial Boundary Condition

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Abstract

We consider the exterior problem for the wave equation. When numerically solving the exterior problem, one often introduces an artificial boundary in order to reduce the computational domain to a bounded domain and imposes an *artificial boundary condition* (ABC) on the artificial boundary. We introduce a new ABC, which is constructed by using the *Dirichlet-to-Neumann* (DtN) operator associated with the Helmholtz equation. Our ABC is suitable for the *controllability method* for computing numerical solutions of the Helmholtz equation. We show the well-posedness of the wave equation with our ABC. Then it is important to investigate some properties of the Hankel functions since the DtN operator on a spherical artificial boundary is analytically represented by the Hankel functions.

1 Introduction

We consider the exterior problem for the wave equation:

$$(1)\qquad \begin{cases} u_{tt} - \Delta u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \gamma, \end{cases}$$

where Ω is an unbounded domain of $\boldsymbol{R}^d$ ($d = 2$ or 3) with boundary γ of class C^∞. We assume that $\mathcal{O} = \boldsymbol{R}^d \setminus \overline{\Omega}$ is a bounded open set. If $f(x,\,t) = F(x)e^{-ikt}$, where k is a positive constant, and if F has a compact support, then the *limiting amplitude principle* holds, that is, the solution $u(x,\,t)$ converges locally to a steady state $U(x)e^{-ikt}$ as time tends to infinity, where U is the solution of the Helmholtz equation with the outgoing radiation condition:

$$(2)\qquad \begin{cases} -\Delta U - k^2 U &= F \quad \text{in } \Omega, \\ U &= 0 \quad \text{on } \gamma, \\ r^{\frac{d-1}{2}} \left(\dfrac{\partial U}{\partial r} - ikU \right) &= 0 \quad \text{as } r \longrightarrow \infty, \end{cases}$$

where $r = |x|$ for $x \in \boldsymbol{R}^d$.

When numerically solving the exterior problem for the wave equation, one often introduces an artificial boundary in order to reduce the computational domain to a bounded domain and imposes an *artificial boundary condition* (ABC) on the artificial boundary. We choose the artificial boundary as follows: $\Gamma_a = \{x \in \boldsymbol{R}^d \mid |x| = a\}$, where a is a positive number such that $\overline{\mathcal{O}} \cup \operatorname{supp} F \subset \{x \in \boldsymbol{R}^d \mid |x| < a\}$. Then the bounded computational domain is defined by $\Omega_a = \{x \in \Omega \mid |x| < a\}$.

We introduce a new ABC:

$$(3)\qquad \frac{\partial u}{\partial n} + \frac{\partial u}{\partial t} = -\mathcal{S}u - iku \qquad \text{on } \Gamma_a,$$

where n is the unit normal vector on Γ_a being toward infinity and $\mathcal{S}$ is the *Dirichlet-to-Neumann* (DtN) operator for the Helmholtz equation with the outgoing radiation condition, i.e., $\mathcal{S}$ is defined by the following relation:

$$\frac{\partial U}{\partial n} = -\mathcal{S}U \qquad \text{on } \Gamma_a,$$

where U is the solution of the Helmholtz equation (2). We design our ABC (3) so that $U(x)e^{-ikt}$ can satisfy it.

We consider the wave equation with our ABC:

$$(4)\qquad \begin{cases} u_{tt} - \Delta u &= f & \text{in} \quad Q, \\ u &= 0 & \text{on} \quad \sigma, \\ \dfrac{\partial u}{\partial n} + \dfrac{\partial u}{\partial t} &= -\mathcal{S}u - iku & \text{on} \quad \Sigma, \\ u(x,\,0) &= u_0^*(x) & \text{in} \quad \Omega_a, \\ u_t(x,\,0) &= u_1^*(x) & \text{in} \quad \Omega_a, \end{cases}$$

where $Q = \Omega_a \times (0, \infty)$, $\sigma = \gamma \times (0, \infty)$, and $\Sigma = \Gamma_a \times (0, \infty)$. We expect that if $f(x, t) = F(x)e^{-ikt}$ in the problem (4), then the solution u converges, on Ω_a, to the steady state $U(x)e^{-ikt}$ as time tends to infinity. However it is yet to be proved. In this report, we show the well-posedness of the problem (4), following the way of proof by Ikawa [6]. To accomplish this purpose, we need to investigate some properties of the Hankel functions since the DtN operator on a spherical artificial boundary is analytically represented by the Hankel functions.

We can use our ABC in the numerical technique of Bristeau-Glowinski-Périaux [3], called the *controllability method*, for solving the exterior Helmholtz problem. The use of our ABC makes it possible that we obtain accurate numerical solutions regardless of the size of the artificial boundary (see Koyama [7]). Hence, by using our ABC and by taking a small artificial boundary, we can reduce computational costs.

An ABC introduced by Engquist-Halpern [4] motivated us to consider our ABC. They consider the case where the force term f of the wave equation (1) depends only on the space variable x. Their ABC is given as follows:

$$\frac{\partial u}{\partial n} + \frac{\partial u}{\partial t} = -\mathcal{T} u \quad \text{on } \Gamma_a,$$

where $\mathcal{T}$ is the DtN operator for the Laplace equation. Their ABC forces the solution of the wave equation to converge, on Ω_a, to a solution of the Laplace equation as time tends to infinity.

This report is organized as follows. In Section 2, we define the Sobolev space on Γ_a, i.e., $H^s(\Gamma_a)$ ($s \in \boldsymbol{R}$), and show that the DtN operator $\mathcal{S}$ is a bounded linear operator from $H^{1/2}(\Gamma_a)$ into $H^{-1/2}(\Gamma_a)$. In Section 3, we state main theorem and prove it. In Appendix A, we describe some properties of the Hankel functions. In Appendix B, we study the Poisson equation with the following ABC:

$$\frac{\partial u}{\partial n} = -\mathcal{T} u \quad \text{on } \Gamma_a.$$

2 Properties of the DtN Operator

We denote by $L^2(\Gamma_a)$ the usual space of complex-valued square integrable functions on Γ_a. Let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|_{L^2(\Gamma_a)}$ denote the inner product and the norm of $L^2(\Gamma_a)$, respectively. We define the Sobolev space $H^s(\Gamma_a)$ in the following. We first describe the two-dimensional case. The polar coordinates are denoted by r, θ. The spherical harmonics Y_n ($n \in \boldsymbol{Z}$) are defined by $Y_n(\theta) = e^{in\theta}/\sqrt{2\pi a}$. Then $\{Y_n \mid n \in \boldsymbol{Z}\}$ becomes a complete orthonormal system of $L^2(\Gamma_a)$. For every $\varphi \in L^2(\Gamma_a)$, we denote the Fourier coefficients of φ by $\varphi_n = \langle \varphi, Y_n \rangle$ ($n \in \boldsymbol{Z}$). For each $s > 0$, we define $H^s(\Gamma_a)$ by

$$H^s(\Gamma_a) = \left\{ \varphi \in L^2(\Gamma_a) \mid \sum_{n=-\infty}^{\infty} |n|^{2s} |\varphi_n|^2 < \infty \right\}.$$

Then $H^s(\Gamma_a)$ becomes a Hilbert space equipped with the following inner product:

$$\langle \varphi, \psi \rangle_{H^s(\Gamma_a)} = \sum_{n=-\infty}^{\infty} (1 + |n|^{2s}) \varphi_n \overline{\psi_n} \quad \text{for all } \varphi,\ \psi \in H^s(\Gamma_a).$$

For each $s < 0$, the space $H^s(\Gamma_a)$ is the dual space of $H^{-s}(\Gamma_a)$, and for $s = 0$, the space $H^0(\Gamma_a) = L^2(\Gamma_a)$.

We next describe the three-dimensional case. The spherical coordinates are denoted by r, θ, ϕ. The spherical harmonics Y_n^m $(n \in \boldsymbol{N} \cup \{0\},\ -n \le m \le n)$ are defined by

$$Y_n^m(\theta,\,\phi) = \frac{1}{a}\sqrt{\frac{(2n+1)}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta)e^{im\phi},$$

where P_n^m are the associated Legendre functions. Then $\{Y_n^m \mid n \in \boldsymbol{N} \cup \{0\},\ -n \le m \le n\}$ is a complete orthonormal system of $L^2(\Gamma_a)$. We denote the Fourier coefficients of $\varphi \in L^2(\Gamma_a)$ by $\varphi_n^m = \langle \varphi,\, Y_n^m\rangle$. For each $s > 0$, we define $H^s(\Gamma_a)$ by

$$H^s(\Gamma_a) = \left\{\varphi \in L^2(\Gamma_a) \mid \sum_{n=0}^{\infty}\sum_{m=-n}^{n} |n|^{2s}|\varphi_n^m|^2 < \infty\right\},$$

and its inner product by

$$\langle \varphi,\, \psi\rangle_{H^s(\Gamma_a)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(1+|n|^{2s})\varphi_n^m\overline{\psi_n^m} \qquad \text{for all } \varphi,\ \psi \in H^s(\Gamma_a).$$

For every $s \le 0$, we define $H^s(\Gamma_a)$ in the same way as the two-dimensional case.

Proposition **2.1** *The DtN operator $\mathcal{S}$ is a bounded linear operator from $H^{1/2}(\Gamma_a)$ into $H^{-1/2}(\Gamma_a)$.*

Proof. We first note that the DtN operator $\mathcal{S}$ is analytically represented as follows (see Grote-Keller [5]):

$$\mathcal{S}\varphi = \begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} -k\frac{H_n^{(1)'}(ka)}{H_n^{(1)}(ka)}\varphi_n Y_n, & \text{if } d = 2, \\[2ex] \displaystyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n} -k\frac{h_n^{(1)'}(ka)}{h_n^{(1)}(ka)}\varphi_n^m Y_n^m, & \text{if } d = 3, \end{cases}$$

where $H_n^{(1)}$ and $h_n^{(1)}$ are the cylindrical and the spherical Hankel functions of the first kind of order n, respectively, and the prime on functions denotes differentiation with respect to the argument.

Let us begin with the two-dimensional case. For every $\varphi \in H^{1/2}(\Gamma_a)$, we can regard $\mathcal{S}\varphi$ as an element of $H^{-1/2}(\Gamma_a)$ by the following identity:

$$(5) \qquad \langle \mathcal{S}\varphi,\, \psi\rangle = \sum_{n=-\infty}^{\infty} -k\frac{H_n^{(1)'}(ka)}{H_n^{(1)}(ka)}\varphi_n\overline{\psi_n} \qquad \text{for all } \psi \in H^{1/2}(\Gamma_a),$$

where $\langle\cdot,\,\cdot\rangle$ also denotes the duality between $H^{-1/2}(\Gamma_a)$ and $H^{1/2}(\Gamma_a)$. This can be understood in the following way. By the Schwarz inequality, we have

$$(6) \qquad |\langle \mathcal{S}\varphi,\, \psi\rangle| \le \left(\sum_{n=-\infty}^{\infty}\left|\frac{k}{1+|n|}\frac{H_n^{(1)'}(ka)}{H_n^{(1)}(ka)}\right|^2(1+|n|)|\varphi_n|^2\right)^{1/2}\left(\sum_{n=-\infty}^{\infty}(1+|n|)|\psi_n|^2\right)^{1/2}.$$

Here, by the recursion formula:

$$(7) \quad H_\nu^{(1)\prime}(ka) = H_{\nu-1}^{(1)}(ka) - \frac{\nu}{ka} H_\nu^{(1)}(ka) \quad \text{for all } \nu \in \boldsymbol{R}$$

(see Abramowitz-Stegun [1]), we can get

$$(8) \quad k\frac{H_n^{(1)\prime}(ka)}{H_n^{(1)}(ka)} = k\frac{H_{n-1}^{(1)}(ka)}{H_n^{(1)}(ka)} - \frac{n}{a} \quad \text{for all } n \in \boldsymbol{Z}.$$

We here notice that we have the following asymptotic behavior:

$$(9) \quad \frac{H_{\nu-1}^{(1)}(ka)}{H_\nu^{(1)}(ka)} \sim \frac{ka}{2\nu} \quad (\nu \longrightarrow \infty),$$

which will be shown in Lemma A.2 described in Appendix A. By using (8) and (9), we can show that there is a positive constant C such that

$$(10) \quad \left| \frac{k}{1+|n|} \frac{H_n^{(1)\prime}(ka)}{H_n^{(1)}(ka)} \right| \leq C \quad \text{for all } n \in \boldsymbol{Z}.$$

Thus, it follows from (6) and (10) that

$$|\langle \mathcal{S}\varphi, \psi \rangle| \leq C \|\varphi\|_{H^{1/2}(\Gamma_a)} \|\psi\|_{H^{1/2}(\Gamma_a)}.$$

This implies that we can define $\mathcal{S}\varphi$ as an element of $H^{-1/2}(\Gamma_a)$ by (5), and moreover that $\mathcal{S}$ is a bounded operator from $H^{1/2}(\Gamma_a)$ into $H^{-1/2}(\Gamma_a)$.

Let us next consider the three-dimensional case. By (7) and the formula:

$$h_n^{(1)}(ka) = \sqrt{\frac{\pi}{2ka}} H_{n+1/2}^{(1)}(ka),$$

we can get

$$(11) \quad k\frac{h_n^{(1)\prime}(ka)}{h_n^{(1)}(ka)} = k\frac{H_{n-1/2}^{(1)}(ka)}{H_{n+1/2}^{(1)}(ka)} - \frac{n+1}{a} \quad \text{for all } n \in \boldsymbol{N} \cup \{0\}.$$

By using (11) and (9), we can show that $\mathcal{S}$ is a bounded operator from $H^{1/2}(\Gamma_a)$ into $H^{-1/2}(\Gamma_a)$ in the same way as the two-dimensional case. ∎

Now we define a linear operator $\mathcal{B} : H^{1/2}(\Gamma_a) \longrightarrow H^{-1/2}(\Gamma_a)$ as follows: for every $\varphi \in H^{1/2}(\Gamma_a)$,

$$\mathcal{B}\varphi = \begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} -k \operatorname{Re}\left\{ \frac{H_n^{(1)\prime}(ka)}{H_n^{(1)}(ka)} \right\} \varphi_n Y_n, & \text{if } d=2, \\ \displaystyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} -k \operatorname{Re}\left\{ \frac{h_n^{(1)\prime}(ka)}{h_n^{(1)}(ka)} \right\} \varphi_n^m Y_n^m, & \text{if } d=3. \end{cases}$$

PROPOSITION **2.2** *The operator $\mathcal{B}$ is a bounded linear operator from $H^{1/2}(\Gamma_a)$ into $H^{-1/2}(\Gamma_a)$, and satisfies*

$$(12) \quad \langle \mathcal{B}\varphi, \varphi \rangle \geq 0 \quad \textit{for all } \varphi \in H^{1/2}(\Gamma_a).$$

Proof. We can show that $\mathcal{B}$ is a bounded linear operator from $H^{1/2}(\Gamma_a)$ into $H^{-1/2}(\Gamma_a)$ in the same way as the proof of Proposition 2.1.

Now we see from Lemmas A.3 and A.4 that

$$\begin{aligned} -k\,\mathrm{Re}\left\{\frac{H_n^{(1)'}(x)}{H_n^{(1)}(x)}\right\} &> 0 \quad \text{for all } n \in \boldsymbol{Z}, \\ -k\,\mathrm{Re}\left\{\frac{h_n^{(1)'}(x)}{h_n^{(1)}(x)}\right\} &> 0 \quad \text{for all } n \in \boldsymbol{N} \cup \{0\}. \end{aligned}$$

This implies (12). ■

3 Main Results

Let us introduce two spaces:

$$\begin{aligned} V &= \left\{ v \in H^1(\Omega_a) \mid v = 0 \ \text{ on } \gamma \right\}, \\ E &= V \times L^2(\Omega_a), \end{aligned}$$

where for each $m \in \boldsymbol{N}$, $H^m(\Omega_a)$ denotes the usual complex-valued Sobolev space of order m on Ω_a, whose norm is denoted by $\|\cdot\|_{H^m(\Omega_a)}$. The space E becomes a Hilbert space equipped with the following inner product: for $\boldsymbol{u} = \{u_0,\, u_1\}$, $\boldsymbol{v} = \{v_0,\, v_1\} \in E$,

$$(\boldsymbol{u},\, \boldsymbol{v})_E = \int_{\Omega_a} \nabla u_0 \cdot \nabla \overline{v_0}\, dx + \int_{\Omega_a} u_1 \overline{v_1}\, dx + \langle \mathcal{B} u_0,\, v_0 \rangle .$$

We denote the associated norm by $\|\cdot\|_E$.

To transform the wave equation to a system of first order, we define a linear operator $\mathcal{A} : D(\mathcal{A})(\subset E) \longrightarrow E$ as follows:

$$\mathcal{A}\boldsymbol{v} = \{v_1,\, \Delta v_0\} \quad \text{for all } \boldsymbol{v} = \{v_0,\, v_1\} \in D(\mathcal{A}),$$

where

$$D(\mathcal{A}) = \Big\{ \boldsymbol{v} = \{v_0,\, v_1\} \mid v_0 \in H^2(\Omega_a) \cap V,\ v_1 \in V, \quad \frac{\partial v_0}{\partial n} + v_1 = -\mathcal{S} v_0 - ikv_0 \ \text{ on } \Gamma_a \Big\}.$$

The problem (4) is written as follows:

$$\begin{cases} \dfrac{d\boldsymbol{u}}{dt}(t) = \mathcal{A}\boldsymbol{u}(t) + \boldsymbol{f}(t) & \text{for } t \in (0,\, \infty), \\ \boldsymbol{u}(0) = \boldsymbol{u}^*, \end{cases}$$

where $\boldsymbol{f}(t) = \{0,\, f(t)\}$ and $\boldsymbol{u}^* = \{u_0^*,\, u_1^*\}$.

Main theorem of this report is the following.

Theorem **3.1** *The linear operator $\mathcal{A}$ is the infinitesimal generator of a semigroup of class C_0.*

In order to prove this theorem, it suffices from Hille-Yosida's theorem to prove three propositions described below (see [6]).

PROPOSITION **3.1** *$D(\mathcal{A})$ is dense in E.*

PROPOSITION **3.2** *There is a positive constant C such that*

(13) $\mathrm{Re}(\boldsymbol{\mathcal{A}u}, \boldsymbol{u})_E \le C\|\boldsymbol{u}\|_E^2$ *for all $\boldsymbol{u} \in D(\mathcal{A})$.*

PROPOSITION **3.3** *For every $\lambda \ge 0$, there exists $(\lambda - \mathcal{A})^{-1}$.*

Propositions 3.1–3.3 will be proved in Subsections 3.1–3.3, respectively.

3.1 Proof of Proposition 3.1

To prove Proposition 3.1, we first prove two lemmas.

LEMMA **3.1** *The space $C_0^\infty(\Omega_a \cup \Gamma_a)$ is dense in V equipped with the induced topology from $H^1(\Omega_a)$, where*

$$C_0^\infty(\Omega_a \cup \Gamma_a) = \{\varphi \in C^\infty(\Omega_a) \mid \text{ There is a } \tilde{\varphi} \in C_0^\infty(\Omega) \text{ such that } \varphi = \tilde{\varphi}|_{\Omega_a}\}.$$

Proof. For every $v \in V$, there is a $\tilde{v} \in H_0^1(\Omega)$ such that $v = \tilde{v}|_{\Omega_a}$. Then there are $\tilde{\varphi}_k \in C_0^\infty(\Omega)$ $(k = 1, 2, \ldots)$ such that $\tilde{\varphi}_k \longrightarrow \tilde{v}$ in $H^1(\Omega)$ as $k \longrightarrow \infty$. We here set $\varphi_k = \tilde{\varphi}_k|_{\Omega_a}$. Then $\varphi_k \in C_0^\infty(\Omega_a \cup \Gamma_a)$ and $\varphi_k \longrightarrow v$ in $H^1(\Omega_a)$ as $k \longrightarrow \infty$. ■

LEMMA **3.2** *For all $g \in C^\infty(\Gamma_a)$ and for all $\varepsilon > 0$, there is a $u \in C_0^\infty(\Omega_a \cup \Gamma_a)$ such that*

$$u = 0 \quad \text{on } \Gamma_a, \qquad \frac{\partial u}{\partial n} = g \quad \text{on } \Gamma_a,$$

and $\|u\|_{H^1(\Omega_a)} \le \varepsilon$.

Proof. A proof is written in [6]. However, we rewrite it in more detail.
We take $\chi(r) \in C_0^\infty(\boldsymbol{R})$ such that

$$\chi(0) = 0, \quad \frac{d\chi}{dr}(0) = 1, \quad \text{and} \quad \mathrm{supp}\,\chi \subset [-1, 1].$$

We choose a neighborhood of Γ_a as follows: $U_\alpha = \{x = r\omega \mid |r - a| \le \alpha,\ \omega \in S^{d-1}\}$, where α is a sufficiently small positive number and S^{d-1} is a unit sphere in $\boldsymbol{R}^d$ with center at the origin. For $\eta \in (0, \alpha)$, we define

$$u_\eta(x) = \begin{cases} g(\omega)\eta\chi\left(\dfrac{r-a}{\eta}\right) & \text{on} \quad U_\alpha, \\ 0 & \text{on} \quad \boldsymbol{R}^d \setminus U_\alpha. \end{cases}$$

Then we have $u_\eta|_{\Omega_a} \in C_0^\infty(\Omega_a \cup \Gamma_a)$,

(14) $u_\eta = 0 \quad \text{on } \Gamma_a, \quad \text{and} \quad \dfrac{\partial u_\eta}{\partial n} = g \quad \text{on } \Gamma_a.$

We further have

$$\int_{\Omega_a} |u_\eta|^2 \, dx \leq \int_{S^{d-1}} \int_{a-\eta}^{a} \left| g(\omega)\eta\chi\left(\frac{r-a}{\eta}\right)\right|^2 r^{d-1} dr \, d\omega.$$

Setting $\rho = (r-a)/\eta$, we can get

$$\int_{\Omega_a} |u_\eta|^2 \, dx \leq \int_{S^{d-1}} \int_{-1}^{0} |g(\omega)\eta\chi(\rho)|^2 (\rho\eta + a)^{d-1} \eta d\rho \, d\omega.$$

Thus there is a positive constant C such that, for sufficiently small $\eta > 0$,

$$(15) \quad \int_{\Omega_a} |u_\eta|^2 \, dx \leq C\eta^3.$$

We next show that there is a positive constant C such that, for sufficiently small $\eta > 0$,

$$(16) \quad \int_{\Omega_a} |\nabla u_\eta|^2 \, dx \leq C\eta.$$

We prove (16) only in the two-dimensional case. Indeed, we have

$$\begin{aligned}
&\int_{\Omega_a} |\nabla u_\eta|^2 \, dx \\
&= \int_{a-\eta}^{a} \int_{S^1} \left\{ \left| g(\omega)\eta \frac{\partial}{\partial r}\chi\left(\frac{r-a}{\eta}\right)\right|^2 + \frac{1}{r^2}\left|\frac{\partial}{\partial\theta} g(\omega)\eta\chi\left(\frac{r-a}{\eta}\right)\right|^2 \right\} r \, d\omega dr \\
&= \eta \int_{-1}^{0} \int_{S^1} |g(\omega)\chi'(\rho)|^2 (\eta\rho + a) \, d\omega d\rho + \eta^3 \int_{-1}^{0} \int_{S^1} \left|\frac{\partial}{\partial\theta} g(\omega)\chi(\rho)\right|^2 \frac{1}{\eta\rho + a} \, d\omega d\rho.
\end{aligned}$$

This yields (16) for sufficiently small η. We can analogously show (16) in the three-dimensional case. It follows from (15) and (16) that if we take η sufficiently small, we can make $\|u_\eta\|_{H^1(\Omega_a)}$ small. This fact and (14) complete the proof of Lemma 3.2. ■

Proof of Proposition 3.1. Let $\{v_0, v_1\}$ be an arbitrary element of E. We can see from Lemma 3.1 that there are $v_{0j} \in C_0^\infty(\Omega_a \cup \Gamma_a)$ $(j = 1, 2, \ldots)$ such that

$$(17) \quad v_{0j} \longrightarrow v_0 \quad \text{in } V \quad \text{as } j \longrightarrow \infty.$$

Since $C_0^\infty(\Omega_a \cup \Gamma_a)$ is also dense in $L^2(\Omega_a)$, there are $v_{1j} \in C_0^\infty(\Omega_a \cup \Gamma_a)$ $(j = 1, 2, \ldots)$ such that

$$(18) \quad v_{1j} \longrightarrow v_1 \quad \text{in } L^2(\Omega_a) \quad \text{as } j \longrightarrow \infty.$$

We here define

$$(19) \quad g_j = \frac{\partial v_{0j}}{\partial n} + v_{1j} + \mathcal{S}v_{0j} + ikv_{0j} \quad \text{on } \Gamma_a.$$

Since $g_j \in C^\infty(\Gamma_a)$, we can see from Lemma 3.2 that for each $j \in \boldsymbol{N}$, there is a $w_j \in C_0^\infty(\Omega_a \cup \Gamma_a)$ such that

$$(20) \quad w_j = 0 \quad \text{on } \Gamma_a, \qquad \frac{\partial w_j}{\partial n} = g_j \quad \text{on } \Gamma_a,$$

and

(21) $\|w_j\|_{H^1(\Omega_a)} \le \dfrac{1}{j}.$

Then we have $\{v_{0j} - w_j,\, v_{1j}\} \in D(\mathcal{A})$. Indeed, it is obvious that $v_{0j} - w_j \in H^2(\Omega_a) \cap V$ and $v_{1j} \in V$. By (20) and (19), we have

$$\begin{aligned}
&\frac{\partial}{\partial n}(v_{0j} - w_j) + v_{1j} + \mathcal{S}(v_{0j} - w_j) + ik(v_{0j} - w_j) \\
= \;& \frac{\partial v_{0j}}{\partial n} + v_{1j} + \mathcal{S}v_{0j} + ikv_{0j} - \frac{\partial w_j}{\partial n} = g_j - g_j = 0 \qquad \text{on } \Gamma_a.
\end{aligned}$$

Furthermore, it follows from (17), (18), and (21) that

$$\{v_{0j} - w_j,\, v_{1j}\} \longrightarrow \{v_0,\, v_1\} \quad \text{in } E \quad \text{as } j \longrightarrow \infty. \quad \blacksquare$$

3.2 Proof of Proposition 3.2

Proof of Proposition 3.2. For every $\boldsymbol{u} = \{u_0,\, u_1\} \in D(\mathcal{A})$, by the Green formula, we have

$$\begin{aligned}
&(\mathcal{A}\boldsymbol{u},\, \boldsymbol{u})_E \\
= \;& \int_{\Omega_a} \nabla u_1 \cdot \nabla \overline{u_0}\, dx + \int_{\Omega_a} \Delta u_0 \overline{u_1}\, dx + \langle \mathcal{B}u_1,\, u_0\rangle \\
= \;& \int_{\Omega_a} \nabla u_1 \cdot \nabla \overline{u_0}\, dx + \int_{\Gamma_a} \frac{\partial u_0}{\partial n}\overline{u_1}\, d\gamma - \int_{\Omega_a} \nabla u_0 \cdot \nabla \overline{u_1}\, dx + \langle \mathcal{B}u_1,\, u_0\rangle \\
= \;& \int_{\Omega_a} \nabla u_1 \cdot \nabla \overline{u_0}\, dx - \int_{\Omega_a} \nabla u_0 \cdot \nabla \overline{u_1}\, dx \\
& - \langle u_1,\, u_1\rangle - \langle (\mathcal{S} - \mathcal{B})u_0,\, u_1\rangle - ik\,\langle u_0,\, u_1\rangle - \langle \mathcal{B}u_0,\, u_1\rangle + \langle \mathcal{B}u_1,\, u_0\rangle\,.
\end{aligned}$$

The real part of this identity is:

(22) $\mathrm{Re}(\mathcal{A}\boldsymbol{u},\, \boldsymbol{u})_E = -\|u_1\|^2_{L^2(\Gamma_a)} - \mathrm{Re}\,\langle (\mathcal{S} - \mathcal{B})u_0,\, u_1\rangle + k\,\mathrm{Im}\,\langle u_0,\, u_1\rangle\,.$

We first consider the case when $d = 2$. We set $\varphi = u_0|_{\Gamma_a}$ and $\psi = u_1|_{\Gamma_a}$. Then we have

(23) $-\mathrm{Re}\,\langle (\mathcal{S} - \mathcal{B})u_0,\, u_1\rangle + k\,\mathrm{Im}\,\langle u_0,\, u_1\rangle = \displaystyle\sum_{n=-\infty}^{\infty} k\left[1 - \mathrm{Im}\left\{\frac{H_n^{(1)'}(ka)}{H_n^{(1)}(ka)}\right\}\right] \mathrm{Im}\left(\varphi_n \overline{\psi_n}\right).$

Now it follows from Lemma A.5 that there is a positive constant C_0 such that

(24) $k\left|1 - \mathrm{Im}\left\{\dfrac{H_n^{(1)'}(ka)}{H_n^{(1)}(ka)}\right\}\right| \le k\left(1 + \mathrm{Im}\left\{\dfrac{H_0^{(1)'}(ka)}{H_0^{(1)}(ka)}\right\}\right) \equiv C_0 \quad$ for all $n \in \boldsymbol{Z}$.

Combining (22), (23), and (24), we have

$$\begin{aligned}
\mathrm{Re}(\mathcal{A}\boldsymbol{u},\, \boldsymbol{u})_E \;&\le\; -\|u_1\|^2_{L^2(\Gamma_a)} + C_0\|u_0\|_{L^2(\Gamma_a)}\|u_1\|_{L^2(\Gamma_a)} \\
&\le\; -\frac{1}{2}\|u_1\|^2_{L^2(\Gamma_a)} + \frac{C_0^2}{2}\|u_0\|^2_{L^2(\Gamma_a)} \le \frac{C_0^2}{2}\|u_0\|^2_{L^2(\Gamma_a)}.
\end{aligned}$$

Further, by using the trace theorem and the Poincaré inequality, we can get (13).

When $d = 3$, by using Lemma A.6 instead of Lemma A.5, we can show (13) in the same way as the case when $d = 2$. $\blacksquare$

3.3 Proof of Proposition 3.3

For $\lambda \in \boldsymbol{R}$ and $\boldsymbol{f} \in E$, we suppose that $\boldsymbol{u} \in D(\mathcal{A})$ satisfies

$$(\lambda - \mathcal{A})\boldsymbol{u} = \boldsymbol{f}.$$

Then we have

$$(25)\quad \begin{cases} -\Delta u_0 + \lambda^2 u_0 = f_1 + \lambda f_0 & \text{in} \quad \Omega_a, \\ u_0 = 0 & \text{on} \quad \gamma, \\ \dfrac{\partial u_0}{\partial n} = -\mathcal{S}u_0 - iku_0 - \lambda u_0 + f_0 & \text{on} \quad \Gamma_a, \end{cases}$$

and

$$(26)\quad u_1 = \lambda u_0 - f_0.$$

Hence, to prove Proposition 3.3, we consider the following problem:

$$(27)\quad \begin{cases} -\Delta u + \lambda^2 u = f & \text{in} \quad \Omega_a, \\ u = 0 & \text{on} \quad \gamma, \\ \dfrac{\partial u}{\partial n} = -\mathcal{S}u - iku - \lambda u + g & \text{on} \quad \Gamma_a, \end{cases}$$

and prove the following proposition, which plays an essential role in the proof of Proposition 3.3.

Proposition **3.4** *For each $\lambda \geq 0$, for every $f \in L^2(\Omega_a)$, and for every $g \in H^{1/2}(\Gamma_a)$, the problem (27) has a unique solution which belongs to $H^2(\Omega_a)$.*

We now consider the following problem:

$$(28)\quad \begin{cases} -\Delta u = f & \text{in} \quad \Omega_a, \\ u = 0 & \text{on} \quad \gamma, \\ \dfrac{\partial u}{\partial n} + \mathcal{T}u = g & \text{on} \quad \Gamma_a, \end{cases}$$

where $\mathcal{T}$ is the DtN operator associated with the Laplace equation and can be represented as follows:

$$\mathcal{T}\varphi = \begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} \frac{|n|}{a}\varphi_n Y_n, & \text{if } d = 2, \\ \displaystyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{n+1}{a}\varphi_n^m Y_n^m, & \text{if } d = 3. \end{cases}$$

It is easily seen that $\mathcal{T}$ is a bounded linear operator form $H^{1/2}(\Gamma_a)$ into $H^{-1/2}(\Gamma_a)$ and satisfies

$$(29)\quad \langle \mathcal{T}\varphi, \varphi\rangle \geq 0 \qquad \text{for all } \varphi \in H^{1/2}(\Gamma_a).$$

Lemma **3.3** *For all $f \in L^2(\Omega_a)$ and for all $g \in H^{1/2}(\Gamma_a)$, the problem (28) has a unique solution belonging to $H^2(\Omega_a)$, and we have the following a priori estimate*

$$(30)\quad \|u\|_{H^2(\Omega_a)} \le C\left\{\|f\|_{L^2(\Omega_a)} + \|g\|_{H^{1/2}(\Gamma_a)}\right\}.$$

We shall prove Lemma 3.3 in Appendix B.

For $\varepsilon \ge 0$, we consider the following problem:

$$(P_\varepsilon)\begin{cases} L_\varepsilon u \equiv -\Delta u + \varepsilon\lambda^2 u &= f \quad \text{in} \quad \Omega_a, \\ u &= 0 \quad \text{on} \quad \gamma, \\ K_\varepsilon u \equiv \dfrac{\partial u}{\partial n} + \mathcal{T}u + \varepsilon\mathcal{R}u &= g \quad \text{on} \quad \Gamma_a, \end{cases}$$

where $\mathcal{R} = \mathcal{S} - \mathcal{T} + ik + \lambda$.

Lemma **3.4** *Let $\lambda \ge 0$ and $0 \le \varepsilon \le 1$. Assume that $u \in H^2(\Omega_a)$ satisfies (P_ε) with $f \in L^2(\Omega_a)$ and $g \in H^{1/2}(\Gamma_a)$. Then there is a positive constant C such that*

$$(31)\quad \|u\|_{H^1(\Omega_a)} \le C\left\{\|f\|_{L^2(\Omega_a)} + \|g\|_{L^2(\Gamma_a)}\right\},$$

where C is independent of λ, ε, f, g, and u.

Proof. By the Green formula, we can get

$$\begin{aligned} &\int_{\Omega_a} |\nabla u|^2\,dx + \varepsilon\lambda^2\int_{\Omega_a}|u|^2\,dx + \langle \mathcal{T}u,\, u\rangle + \varepsilon\{\langle(\mathcal{S}-\mathcal{T})u,\, u\rangle + ik\,\langle u,\, u\rangle + \lambda\,\langle u,\, u\rangle\} \\ &= \int_{\Omega_a} f\overline{u}\,dx + \langle g,\, u\rangle. \end{aligned}$$

The real part of this identity is:

$$\begin{aligned} &\int_{\Omega_a} |\nabla u|^2\,dx + \varepsilon\lambda^2\int_{\Omega_a}|u|^2\,dx + (1-\varepsilon)\,\langle \mathcal{T}u,\, u\rangle + \varepsilon\{\mathrm{Re}\,\langle\mathcal{S}u,\, u\rangle + \lambda\|u\|^2_{L^2(\Gamma_a)}\} \\ &= \mathrm{Re}\int_{\Omega_a} f\overline{u}\,dx + \mathrm{Re}\,\langle g,\, u\rangle. \end{aligned}$$

We here note that we have

$$(32)\quad \mathrm{Re}\,\langle\mathcal{S}u,\, u\rangle = \langle\mathcal{B}u,\, u\rangle \ge 0.$$

Thus it follows from (29), (32), and the conditions of ε and λ that

$$\int_{\Omega_a}|\nabla u|^2\,dx \le \mathrm{Re}\int_{\Omega_a} f\overline{u}\,dx + \mathrm{Re}\,\langle g,\, u\rangle.$$

Therefore, by the Poincaré inequality and the trace theorem, we obtain (31). ■

Lemma **3.5** *Let λ be an arbitrary non-negative number. Then there is an $\alpha > 0$ such that if, for an $\varepsilon_1 \in [0, 1]$, the problem (P_{ε_1}) has a solution belonging to $H^2(\Omega_a)$ for every $f \in L^2(\Omega_a)$ and for every $g \in H^{1/2}(\Gamma_a)$, then, for each ε satisfying $|\varepsilon - \varepsilon_1| < \alpha$, the problem (P_ε) has a solution belonging to $H^2(\Omega_a)$ for every $f \in L^2(\Omega_a)$ and for every $g \in H^{1/2}(\Gamma_a)$.*

Proof. For every $f \in L^2(\Omega_a)$ and for every $g \in H^{1/2}(\Gamma_a)$, let $u^{(0)}$ be a solution of the problem (P_{ε_1}), which belongs to $H^2(\Omega_a)$. For $p = 0, 1, 2, \ldots$, let $u^{(p+1)} \in H^2(\Omega_a)$ be a solution of the following problem:

$$\left\{\begin{array}{rcll} L_{\varepsilon_1} u^{(p+1)} & = & (\varepsilon_1 - \varepsilon)\lambda^2 u^{(p)} & \text{in} \quad \Omega_a, \\ u^{(p+1)} & = & 0 & \text{on} \quad \gamma, \\ K_{\varepsilon_1} u^{(p+1)} & = & (\varepsilon_1 - \varepsilon)\mathcal{R} u^{(p)} & \text{on} \quad \Gamma_a. \end{array}\right.$$

Then, by Lemma 3.4, we have, for every $p \in \boldsymbol{N} \cup \{0\}$,

$$(33) \quad \|u^{(p+1)}\|_{H^1(\Omega_a)} \leq C_1 \left\{ |\varepsilon_1 - \varepsilon|\lambda^2 \|u^{(p)}\|_{L^2(\Omega_a)} + |\varepsilon_1 - \varepsilon| \|\mathcal{R} u^{(p)}\|_{L^2(\Gamma_a)} \right\},$$

where C_1 is a positive constant independent of λ, ε, ε_1, $u^{(p)}$, and $u^{(p+1)}$. We here note that Lemmas A.7 and A.8 imply that $\mathcal{R}$ is a bounded linear operator on $H^s(\Gamma_a)$ for every $s \in \boldsymbol{R}$. Thus, it follows from (33) and the trace theorem that there is a positive constant C_2 such that

$$\|u^{(p+1)}\|_{H^1(\Omega_a)} \leq C_2 |\varepsilon_1 - \varepsilon| \|u^{(p)}\|_{H^1(\Omega_a)},$$

where C_2 is independent of ε, ε_1, $u^{(p)}$, and $u^{(p+1)}$. This yields

$$(34) \quad \|u^{(p+1)}\|_{H^1(\Omega_a)} \leq (C_2 |\varepsilon_1 - \varepsilon|)^{p+1} \|u^{(0)}\|_{H^1(\Omega_a)}.$$

We here set $u_q = \sum_{p=0}^{q} u^{(p)}$. Then we can see from (34) that if $C_2|\varepsilon_1 - \varepsilon| < 1$, then $\{u_q\}_{q=1}^{\infty}$ is a Cauchy sequence in V. Furthermore we note that we have

$$(35) \quad \left\{\begin{array}{rcll} L_{\varepsilon_1} u_q & = & (\varepsilon_1 - \varepsilon)\lambda^2 u_{q-1} + f & \text{in} \quad \Omega_a, \\ u_q & = & 0 & \text{on} \quad \gamma, \\ K_{\varepsilon_1} u_q & = & (\varepsilon_1 - \varepsilon)\mathcal{R} u_{q-1} + g & \text{on} \quad \Gamma_a, \end{array}\right.$$

and hence, for $q > q'$,

$$\left\{\begin{array}{rcll} -\Delta(u_q - u_{q'}) & = & f_{qq'} & \text{in} \quad \Omega_a, \\ u_q - u_{q'} & = & 0 & \text{on} \quad \gamma, \\ \dfrac{\partial}{\partial n}(u_q - u_{q'}) + \mathcal{T}(u_q - u_{q'}) & = & g_{qq'} & \text{on} \quad \Gamma_a, \end{array}\right.$$

where

$$\begin{array}{rcl} f_{qq'} & = & -\varepsilon_1 \lambda^2 (u_q - u_{q'}) + (\varepsilon_1 - \varepsilon)\lambda^2 (u_{q-1} - u_{q'-1}), \\ g_{qq'} & = & -\varepsilon_1 \mathcal{R}(u_q - u_{q'}) + (\varepsilon_1 - \varepsilon)\mathcal{R}(u_{q-1} - u_{q'-1}). \end{array}$$

By the a priori estimate (30), we have, for $q > q'$,

$$(36) \quad \|u_q - u_{q'}\|_{H^2(\Omega_a)} \leq C \left\{ \|f_{qq'}\|_{L^2(\Omega_a)} + \|g_{qq'}\|_{H^{1/2}(\Gamma_a)} \right\}.$$

Since $\mathcal{R}$ is a bounded linear operator on $H^{1/2}(\Gamma_a)$, we can see from (36) and the trace theorem that

$$\|u_q - u_{q'}\|_{H^2(\Omega_a)} \leq C \left\{ \|u_q - u_{q'}\|_{H^1(\Omega_a)} + \|u_{q-1} - u_{q'-1}\|_{H^1(\Omega_a)} \right\}.$$

This implies that $\{u_q\}_{q=1}^{\infty}$ is a Cauchy sequence in $H^2(\Omega_a)$ since $\{u_q\}_{q=1}^{\infty}$ is a Cauchy sequence in V. Hence there is a $u \in H^2(\Omega_a)$ such that $u_q \longrightarrow u$ in $H^2(\Omega_a)$. Then we can conclude from (35) that u satisfies

$$-\Delta u + \varepsilon_1 \lambda^2 u = (\varepsilon_1 - \varepsilon)\lambda^2 u + f \quad \text{in } \Omega_a, \qquad u = 0 \quad \text{on } \gamma,$$

$$\frac{\partial u}{\partial n} + \mathcal{T} u + \varepsilon_1 \mathcal{R} u = (\varepsilon_1 - \varepsilon)\mathcal{R} u + g \quad \text{on } \Gamma_a.$$

This shows that u is a solution of the problem (P_ε). We can see from the argument above that if we take $\alpha = 1/C_2$, then the assertion of Lemma 3.5 holds. ■

Proof of Proposition 3.4. Lemma 3.3 assures that the problem (28), i.e., the problem (P_ε) with $\varepsilon = 0$ has a solution belonging to $H^2(\Omega_a)$ for all $f \in L^2(\Omega_a)$ and for all $g \in H^{1/2}(\Gamma_a)$. Thus we can see from Lemma 3.5 that for each $\varepsilon \in (-\alpha, \alpha)$, the problem (P_ε) has a solution belonging to $H^2(\Omega_a)$ for all $\lambda \geq 0$, for all $f \in L^2(\Omega_a)$, and for all $g \in H^{1/2}(\Gamma_a)$. Repeating this argument, we can conclude that for every $\varepsilon \in (-\alpha, 1+\alpha)$, the problem (P_ε) has a solution belonging to $H^2(\Omega_a)$ for all $\lambda \geq 0$, for all $f \in L^2(\Omega_a)$, and for all $g \in H^{1/2}(\Gamma_a)$. Taking here $\varepsilon = 1$, we can see that the problem (27) has a solution belonging to $H^2(\Omega_a)$ for all $\lambda \geq 0$, for all $f \in L^2(\Omega_a)$, and for all $g \in H^{1/2}(\Gamma_a)$. The uniqueness for the solution of the problem (27) follows from (31). ■

Proof of Proposition 3.3. Let λ be an arbitrary non-negative number. We first show that $(\lambda - \mathcal{A})$ is one-to-one. Suppose

$$(\lambda - \mathcal{A})\boldsymbol{u} = 0, \quad \boldsymbol{u} = \{u_0,\, u_1\} \in D(\mathcal{A}),$$

then u_0 satisfies (27) with $f = 0$ and $g = 0$, and hence it follows from Proposition 3.4 that $u_0 = 0$. Further, since by (26) we have $u_1 = \lambda u_0$, we can get $u_1 = 0$.

We next show that $(\lambda - \mathcal{A})$ is onto. We can see from Proposition 3.4 that for every $\boldsymbol{f} = \{f_0,\, f_1\} \in E$, there is a $u_0 \in H^2(\Omega_a)$ such that u_0 satisfies (25). Set $u_1 = \lambda u_0 - f_0 \in V$. Then we can easily see that $\boldsymbol{u} = \{u_0,\, u_1\} \in D(\mathcal{A})$ and $(\lambda - \mathcal{A})\boldsymbol{u} = \boldsymbol{f}$. ■

A Some Properties of the Hankel Functions

LEMMA **A.1** *For each $x > 0$, we have*

$$(37) \quad H_\nu^{(1)}(x) \sim -i\sqrt{\frac{2}{\pi\nu}}\left(\frac{ex}{2\nu}\right)^{-\nu} \qquad (\nu \longrightarrow \infty),$$

where $\nu \in \boldsymbol{R}$.

Proof. According to [1], we have

$$(38) \quad J_\nu(x) \sim \frac{1}{\sqrt{2\pi\nu}}\left(\frac{ex}{2\nu}\right)^{\nu} \qquad (\nu \longrightarrow \infty),$$

$$(39) \quad N_\nu(x) \sim -\sqrt{\frac{2}{\pi\nu}}\left(\frac{ex}{2\nu}\right)^{-\nu} \qquad (\nu \longrightarrow \infty),$$

where J_ν and N_ν are the cylindrical Bessel functions and the cylindrical Neumann functions of order ν, respectively. We have

$$(40)\quad H_\nu^{(1)}(x)\left\{-i\sqrt{\frac{2}{\pi\nu}}\left(\frac{ex}{2\nu}\right)^{-\nu}\right\}^{-1}$$
$$= J_\nu(x)\left\{\frac{1}{\sqrt{2\pi\nu}}\left(\frac{ex}{2\nu}\right)^{\nu}\right\}^{-1}\left\{\frac{1}{\sqrt{2\pi\nu}}\left(\frac{ex}{2\nu}\right)^{\nu}\right\}\left\{-i\sqrt{\frac{2}{\pi\nu}}\left(\frac{ex}{2\nu}\right)^{-\nu}\right\}^{-1}$$
$$+N_\nu(x)\left\{-\sqrt{\frac{2}{\pi\nu}}\left(\frac{ex}{2\nu}\right)^{-\nu}\right\}^{-1}.$$

We here note that

$$(41)\quad \left\{\frac{1}{\sqrt{2\pi\nu}}\left(\frac{ex}{2\nu}\right)^{\nu}\right\}\left\{-i\sqrt{\frac{2}{\pi\nu}}\left(\frac{ex}{2\nu}\right)^{-\nu}\right\}^{-1} = \frac{i}{2}\left(\frac{ex}{2\nu}\right)^{2\nu} \longrightarrow 0 \quad (\nu \longrightarrow \infty).$$

Combining (38) – (41), we can get (37). ■

LEMMA **A.2** *For each $x > 0$, we have*

$$(42)\quad \frac{H_{\nu-1}^{(1)}(x)}{H_\nu^{(1)}(x)} \sim \frac{x}{2\nu} \quad (\nu \longrightarrow \infty),$$

where $\nu \in \boldsymbol{R}$.

Proof. We have

$$(43)\quad \frac{H_{\nu-1}^{(1)}(x)}{H_\nu^{(1)}(x)}\frac{2\nu}{x} = \frac{H_{\nu-1}^{(1)}(x)}{-i\sqrt{\frac{2}{\pi(\nu-1)}}\left(\frac{ex}{2(\nu-1)}\right)^{-(\nu-1)}}\frac{-i\sqrt{\frac{2}{\pi\nu}}\left(\frac{ex}{2\nu}\right)^{-\nu}}{H_\nu^{(1)}(x)}\frac{-i\sqrt{\frac{2}{\pi(\nu-1)}}\left(\frac{ex}{2(\nu-1)}\right)^{-(\nu-1)}}{-i\sqrt{\frac{2}{\pi\nu}}\left(\frac{ex}{2\nu}\right)^{-\nu}}\frac{2\nu}{x}.$$

We here note that

$$(44)\quad \frac{-i\sqrt{\frac{2}{\pi(\nu-1)}}\left(\frac{ex}{2(\nu-1)}\right)^{-(\nu-1)}}{-i\sqrt{\frac{2}{\pi\nu}}\left(\frac{ex}{2\nu}\right)^{-\nu}}\frac{2\nu}{x}$$
$$= \left(1+\frac{1}{\nu-1}\right)^{3/2}\left\{\left(1-\frac{1}{\nu}\right)^{-\nu}\right\}^{-1} e \longrightarrow 1 \quad (\nu \longrightarrow \infty).$$

From (43), (44), and Lemma A.1, we can obtain (42). ■

LEMMA **A.3** *For all $x > 0$ and for all $\nu \in \boldsymbol{R}$, we have*

$$\mathrm{Re}\left\{\frac{H_\nu^{(1)\prime}(x)}{H_\nu^{(1)}(x)}\right\} < 0.$$

Proof. Since $H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x)$, we have

$$(45)\quad \mathrm{Re}\left\{\frac{H_\nu^{(1)'}(x)}{H_\nu^{(1)}(x)}\right\} = \frac{J_\nu(x)J_\nu'(x) + N_\nu(x)N_\nu'(x)}{J_\nu^2(x) + N_\nu^2(x)}.$$

According to Watson [8], we have

$$(46)\quad J_\nu^2(x) + N_\nu^2(x) = \frac{8}{\pi^2}\int_0^\infty K_0(2x\sinh t)\cosh(2\nu t)\,dt,$$

where K_0 is the modified Bessel function of the second kind of order zero. Differentiating (46) with x, we obtain

$$(47)\quad J_\nu(x)J_\nu'(x) + N_\nu(x)N_\nu'(x) = \frac{8}{\pi^2}\int_0^\infty K_0'(2x\sinh t)\sinh t\cosh(2\nu t)\,dt.$$

Now we note that we have the following formula:

$$(48)\quad K_0(\xi) = \int_0^\infty e^{-\xi\cosh t}\,dt \quad \text{for all } \xi > 0 \quad \text{(see Abramowitz and Stegun [1])}.$$

Differentiating (48) with ξ, we can get

$$(49)\quad K_0'(\xi) = -\int_0^\infty e^{-\xi\cosh t}\cosh t\,dt < 0 \quad \text{for all } \xi > 0.$$

Combining (45), (47), and (49) will complete the proof of Lemma A.3. ■

Lemma **A.4** *For all $x > 0$ and for all $n \in \boldsymbol{N} \cup \{0\}$, we have*

$$\mathrm{Re}\left\{\frac{h_n^{(1)'}(x)}{h_n^{(1)}(x)}\right\} < 0.$$

Proof. Since

$$h_n^{(1)}(x) = \sqrt{\frac{\pi}{2x}}H_{n+1/2}^{(1)}(x),$$

we have

$$(50)\quad \frac{h_n^{(1)'}(x)}{h_n^{(1)}(x)} = -\frac{1}{2x} + \frac{H_{n+1/2}^{(1)'}(x)}{H_{n+1/2}^{(1)}(x)}.$$

Thus, we can see form Lemma A.3 that

$$\mathrm{Re}\left\{\frac{h_n^{(1)'}(x)}{h_n^{(1)}(x)}\right\} = -\frac{1}{2x} + \mathrm{Re}\left\{\frac{H_{n+1/2}^{(1)'}(x)}{H_{n+1/2}^{(1)}(x)}\right\} < 0. \quad ■$$

Lemma **A.5** *For all $x > 0$ and for all ν, $\nu' \in \boldsymbol{R}$ satisfying $|\nu| > |\nu'|$, we have*

$$(51)\quad 0 < \mathrm{Im}\left\{\frac{H_\nu^{(1)'}(x)}{H_\nu^{(1)}(x)}\right\} < \mathrm{Im}\left\{\frac{H_{\nu'}^{(1)'}(x)}{H_{\nu'}^{(1)}(x)}\right\}.$$

Proof. We have the following formulas:

$$H_\nu^{(1)'}(x) = H_{\nu-1}^{(1)}(x) - \frac{\nu}{x} H_\nu^{(1)}(x),$$

$$J_{\nu-1}(x)N_\nu(x) - J_\nu(x)N_{\nu-1}(x) = -\frac{2}{\pi x} \quad \text{(see [1])}.$$

Using these formulas, we can get

$$\text{(52)} \quad \mathrm{Im}\left\{\frac{H_\nu^{(1)'}(x)}{H_\nu^{(1)}(x)}\right\} = \frac{2}{\pi x}\frac{1}{J_\nu^2(x) + N_\nu^2(x)} > 0.$$

Now it follows from (48) that $K_0(2x \sinh t)$, being in the integral on the right-hand side of (46), is a positive function of t on $(0, \infty)$. Thus, we can easily see from (46) that for all ν, $\nu' \in \boldsymbol{R}$ satisfying $|\nu| > |\nu'|$,

$$\text{(53)} \quad J_\nu^2(x) + N_\nu^2(x) > J_{\nu'}^2(x) + N_{\nu'}^2(x).$$

From (52) and (53), we can get (51). ∎

Lemma **A.6** *For all $x > 0$ and for all $n \in \boldsymbol{N}$, we have*

$$0 < \mathrm{Im}\left\{\frac{h_n^{(1)'}(x)}{h_n^{(1)}(x)}\right\} < \mathrm{Im}\left\{\frac{h_0^{(1)'}(x)}{h_0^{(1)}(x)}\right\} \equiv 1.$$

Proof. From (50), we can get

$$\mathrm{Im}\left\{\frac{h_n^{(1)'}(x)}{h_n^{(1)}(x)}\right\} = \mathrm{Im}\left\{\frac{H_{n+1/2}^{(1)'}(x)}{H_{n+1/2}^{(1)}(x)}\right\}.$$

Thus, by Lemma A.5, we have

$$0 < \mathrm{Im}\left\{\frac{h_n^{(1)'}(x)}{h_n^{(1)}(x)}\right\} < \mathrm{Im}\left\{\frac{h_0^{(1)'}(x)}{h_0^{(1)}(x)}\right\} \qquad \text{for all } n \in \boldsymbol{N}.$$

Since $h_0^{(1)}(x) = -ie^{ix}/x$, we can see that

$$\mathrm{Im}\left\{\frac{h_0^{(1)'}(x)}{h_0^{(1)}(x)}\right\} \equiv 1. \quad ∎$$

Lemma **A.7** *Let $k > 0$ and $a > 0$. Then there is a positive constant C such that*

$$\text{(54)} \quad \left| k\frac{H_n^{(1)'}(ka)}{H_n^{(1)}(ka)} + \frac{|n|}{a} \right| \leq C \qquad \textit{for all } n \in \boldsymbol{Z}.$$

Proof. From Lemma A.2, we can see that there is a positive constant C such that

$$\left| \frac{H_{n-1}^{(1)}(ka)}{H_n^{(1)}(ka)} \frac{2n}{ka} \right| \le C \quad \text{for all } n \in \boldsymbol{N} \cup \{0\}.$$

This implies that

$$\left| \frac{H_{n-1}^{(1)}(ka)}{H_n^{(1)}(ka)} \right| \le C \frac{ka}{2n} \quad \text{for all } n \in \boldsymbol{N}.$$

Thus, by using (8), we can get

$$\left| k \frac{H_n^{(1)\prime}(ka)}{H_n^{(1)}(ka)} + \frac{n}{a} \right| \le C \frac{k^2 a}{2n} \quad \text{for all } n \in \boldsymbol{N}.$$

Further, noting that $H_{-n}^{(1)}(ka) = (-1)^n H_n^{(1)}(ka)$, we see that (54) holds. ■

LEMMA **A.8** *Let $k > 0$ and $a > 0$. Then there is a positive constant C such that*

$$\left| k \frac{h_n^{(1)\prime}(ka)}{h_n^{(1)}(ka)} + \frac{n+1}{a} \right| \le C \quad \textit{for all } n \in \boldsymbol{N} \cup \{0\}. \tag{55}$$

Proof. We can see from (42) that there is a positive constant C such that

$$\left| \frac{H_{n-1/2}^{(1)}(ka)}{H_{n+1/2}^{(1)}(ka)} \right| \le C \frac{ka}{2n+1} \quad \text{for all } n \in \boldsymbol{N} \cup \{0\}. \tag{56}$$

From (11) and (56), we can get (55). ■

B Proof of Lemma 3.3

In the proof of Lemma 3.3, it is essential to prove the following theorem.

THEOREM **B.1** *For every $f \in L^2(\Omega_a)$, there is a unique function $u \in H^2(\Omega_a)$ such that*

$$\begin{cases} -\Delta u &= f & \textit{in} & \Omega_a, \\ u &= 0 & \textit{on} & \gamma, \\ \dfrac{\partial u}{\partial n} &= -\mathcal{T} u & \textit{on} & \Gamma_a. \end{cases} \tag{57}$$

Further there is a positive constant C such that

$$\|u\|_{H^2(\Omega_a)} \le C \|f\|_{L^2(\Omega_a)}, \tag{58}$$

where C is independent of u and f.

If we prove Theorem B.1, then we can easily prove Lemma 3.3 by virtue of the following lemma.

Lemma **B.1** *For every $g \in H^{1/2}(\Gamma_a)$, there is a $w \in H^2(\Omega_a) \cap V$ such that*

$$(59) \quad \frac{\partial w}{\partial n} + \mathcal{T} w = g \qquad \text{on } \Gamma_a.$$

Further there is a positive constant C such that

$$(60) \quad \|w\|_{H^2(\Omega_a)} \leq C \|g\|_{H^{1/2}(\Gamma_a)},$$

where C is independent of w and g.

We shall prove this lemma after describing the proof of Lemma 3.3.

Proof of Lemma 3.3. For every $f \in L^2(\Omega_a)$ and for every $g \in H^{1/2}(\Gamma_a)$, let u be a solution of the problem (28) and w a function satisfying (59) and (60). Setting $v = u - w$, we have

$$-\Delta v = -\Delta u + \Delta w = f + \Delta w \qquad \text{in } \Omega_a,$$

$$v = u - w = 0 \qquad \text{on } \gamma,$$

$$\frac{\partial v}{\partial n} + \mathcal{T} v = \frac{\partial u}{\partial n} + \mathcal{T} u - \left(\frac{\partial w}{\partial n} + \mathcal{T} w \right) = g - g = 0 \qquad \text{on } \Gamma_a.$$

This shows that we can reduce the problem (28) to the problem (57). Thus, the existence and uniqueness for the solution of the problem (28) is assured by Theorem B.1. Further, we can easily get (30) by using (58) and (60). ■

Proof of Lemma B.1. We prove only the two-dimensional case, since we can similarly prove the three-dimensional case. For every $g \in H^{1/2}(\Gamma_a)$, set

$$\varphi = \sum_{|n|>0} \frac{a}{|n|} g_n Y_n,$$

where $g_n = \langle g, Y_n \rangle$. Then we have

$$\left\{ \varphi, \frac{1}{2\pi a} \int_{\Gamma_a} g \, d\gamma \right\} \in H^{3/2}(\Gamma_a) \times H^{1/2}(\Gamma_a).$$

We can see from the trace theorem that there is a $w \in H^2(\Omega_a) \cap V$ such that

$$(61) \quad w = \varphi \ \text{ on } \Gamma_a \quad \text{ and } \quad \frac{\partial w}{\partial n} = \frac{1}{2\pi a} \int_{\Gamma_a} g \, d\gamma \ \text{ on } \Gamma_a,$$

and that there is a positive constant C_1 such that

$$(62) \quad \|w\|_{H^2(\Omega_a)} \leq C_1 \left\{ \|\varphi\|_{H^{3/2}(\Gamma_a)} + \left\| \frac{1}{2\pi a} \int_{\Gamma_a} g \, d\gamma \right\|_{H^{1/2}(\Gamma_a)} \right\},$$

where C_1 is independent of g and w. By (61), we have

$$\frac{\partial w}{\partial n} + \mathcal{T} w = \frac{1}{2\pi a} \int_{\Gamma_a} g \, d\gamma + (g - g_0 Y_0) = g.$$

Next we show (60). By simple calculation, we can get

$$(63) \quad \|\varphi\|_{H^{3/2}(\Gamma_a)} + \left\| \frac{1}{2\pi a} \int_{\Gamma_a} g \, d\gamma \right\|_{H^{1/2}(\Gamma_a)} \leq C_2 \|g\|_{H^{1/2}(\Gamma_a)},$$

where C_2 is a positive constant independent of g. Combining (62) and (63), we can get (60). ■

We shall prove Theorem B.1 in the subsequent subsection.

B.1 Proof of Theorem B.1

To prove Theorem B.1, we consider the exterior problem for the Poisson equation: for f given in $L^2(\Omega)$ which has a compact support, find $u \in W(\Omega)$ such that

$$(64)\quad \begin{cases} -\Delta u &= f \quad \text{in} \quad \Omega, \\ \quad u &= 0 \quad \text{on} \quad \gamma, \end{cases}$$

where $W(\Omega)$ is the weighted Sobolev space defined as follows: if $d = 2$,

$$W(\Omega) = \left\{ u \in \mathcal{D}'(\Omega) \mid \frac{u(x)}{(1+|x|^2)^{1/2}\log(2+|x|^2)} \in L^2(\Omega),\ \frac{\partial u}{\partial x_j} \in L^2(\Omega)\ (j = 1,\, 2) \right\};$$

if $d = 3$,

$$W(\Omega) = \left\{ u \in \mathcal{D}'(\Omega) \mid \frac{u(x)}{(1+|x|^2)^{1/2}} \in L^2(\Omega),\ \frac{\partial u}{\partial x_j} \in L^2(\Omega)\ (j = 1,\, 2,\, 3) \right\},$$

where $\mathcal{D}'(\Omega)$ is the Schwartz space of all distributions on Ω. The weak formulation of the problem (64) is:

$$(65)\quad \begin{cases} \text{Find } u \in W_0(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla \overline{v}\, dx = \int_\Omega f \overline{v}\, dx \quad \text{for all } v \in W_0(\Omega), \end{cases}$$

where $W_0(\Omega) = \{u \in W(\Omega) \mid u = 0 \quad \text{on } \gamma\}$.

Here for every $m \in \boldsymbol{N}$ and for every unbounded domain $\Omega \subset \boldsymbol{R}^d$, we define

$$H^m_{\mathrm{loc}}(\overline{\Omega}) = \{u \mid u \in H^m(U) \ \text{ for all bounded open set } U \subset \Omega\}.$$

THEOREM **B.2** *For every $f \in L^2(\Omega)$ whose support is compact, the problem* (65) *has a unique solution, which belongs to $H^2_{\mathrm{loc}}(\overline{\Omega})$.*

Proof. In Amrouche-Girault-Giroire [2], it is proved that the problem (65) has a unique solution. Form the well-known regularity argument for the solution of the Poisson equation, it follows that the solution belongs to $H^2_{\mathrm{loc}}(\overline{\Omega})$. ■

Proof of Theorem B.1. We first show (58) by assuming that the problem (57) has a unique solution belonging to $H^2(\Omega_a)$ for every $f \in L^2(\Omega_a)$. We define $G : L^2(\Omega_a) \longrightarrow H^2(\Omega_a)$ as follows:

$$Gf = u \quad \text{for all } f \in L^2(\Omega_a),$$

where u is the solution of the problem (57). Then G is a closed operator, and hence, by the closed graph theorem, G is a bounded linear operator from $L^2(\Omega_a)$ into $H^2(\Omega_a)$. This implies (58).

We next show that the problem (57) has a unique solution belonging to $H^2(\Omega_a)$ for every $f \in L^2(\Omega_a)$. If the problem (57) has a solution $u \in H^2(\Omega_a)$, then u is a solution of the following weak formulation:

$$(66)\quad \begin{cases} \text{Find } u \in V \text{ such that} \\ \displaystyle\int_{\Omega_a} \nabla u \cdot \nabla \overline{v}\, dx + \langle \mathcal{T} u,\, v\rangle = \int_{\Omega_a} f \overline{v}\, dx \quad \text{for all } v \in V. \end{cases}$$

Conversely, if the problem (66) has a solution $u \in H^2(\Omega_a)$, then u is a solution of the problem (57). Thus, our task is now to show that the problem (66) has a unique solution belonging to $H^2(\Omega_a)$.

It is easy to show that the problem (66) has a unique solution. Indeed, by virtue of the Poincaré inequality, the space V is a Hilbert space equipped with the inner product:

$$(u,\, v)_V = \int_\Omega \nabla u \cdot \nabla \overline{v}\, dx.$$

The right-hand side of the equality in (66) can be regarded as an element of the dual space of V. Since the operator $\mathcal{T}$ is non-negative, the sesquilinear form on the left-hand side of the equality in (66) is elliptic on V. Therefore it is follows from Lax-Milgram's lemma that the problem (66) has a unique solution u.

It remains to show that the solution u of the problem (66) belongs to $H^2(\Omega_a)$. In order to show this fact, because of Theorem B.2, it is sufficient to show that a harmonic extension of u becomes the solution of the problem (65), namely, to show the following proposition.

Proposition **B.1** *Let u_i be the solution of* (66) *and set $\varphi = u_i|_{\Gamma_a}$. Define a function u on Ω as follows:*

$$u|_{\Omega_a} = u_i, \tag{67}$$

$$u|_{\Omega'_a} = \begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} \left(\frac{a}{r}\right)^{|n|} \varphi_n Y_n(\theta), & \text{if} \quad d = 2, \\ \displaystyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(\frac{a}{r}\right)^{n+1} \varphi_n^m Y_n^m(\theta,\, \phi), & \text{if} \quad d = 3, \end{cases} \tag{68}$$

where $\Omega'_a = \{x \in \boldsymbol{R}^d \mid |x| > a\}$, $\varphi_n = \langle \varphi,\, Y_n \rangle$, and $\varphi_n^m = \langle \varphi,\, Y_n^m \rangle$. Then u is the solution of the problem (65).

To prove Proposition B.1, we shall use the following lemma, whose proof is postponed to the completion of the proof of Proposition B.1.

Lemma **B.2** *Let $\varphi \in H^{1/2}(\Gamma_a)$ and define u, on Ω'_a, by*

$$u = \begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} \left(\frac{a}{r}\right)^{|n|} \varphi_n Y_n(\theta), & \text{if} \quad d = 2, \\ \displaystyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(\frac{a}{r}\right)^{n+1} \varphi_n^m Y_n^m(\theta,\, \phi), & \text{if} \quad d = 3. \end{cases} \tag{69}$$

Then, $u \in W(\Omega'_a)$, that is, the infinite series on the right-hand side of (69) *converge in $W(\Omega'_a)$.*

Proof of Proposition B.1. Let u_i be the solution of the problem (66) and u the function on Ω defined by (67) and (68). We denote $u|_{\Omega'_a}$ by u_e. We first show that $u \in W_0(\Omega)$. Because of Lemma B.2, it suffices to show that $u \in H^1_{\mathrm{loc}}(\overline{\Omega})$. This follows from the following facts: $u_i \in H^1(\Omega_a)$, $u_e \in H^1_{\mathrm{loc}}(\overline{\Omega'_a})$, and $u_i = u_e$ on Γ_a.

We next show that u is the solution of the problem (65). Since $C_0^\infty(\Omega)$ is dense in $W_0(\Omega)$, it suffices to show that

$$(70)\quad -\int_\Omega u\Delta\overline{\psi}\,dx = \int_\Omega f\overline{\psi}\,dx \qquad \text{for all } \psi \in C_0^\infty(\Omega),$$

where f is assumed to be extended by zero to Ω. By the Green formula, we have

$$\begin{aligned} -\int_\Omega u\Delta\overline{\psi}\,dx &= -\int_{\Omega_a} u_i\Delta\overline{\psi}\,dx - \int_{\Omega_a'} u_e\Delta\overline{\psi}\,dx \\ &= -\int_{\Gamma_a} u_i\frac{\partial\overline{\psi}}{\partial r}\,d\gamma + \int_{\Omega_a}\nabla u_i\cdot\nabla\overline{\psi}\,dx + \int_{\Gamma_a} u_e\frac{\partial\overline{\psi}}{\partial r}\,d\gamma + \int_{\Omega_a'}\nabla u_e\cdot\nabla\overline{\psi}\,dx. \end{aligned}$$

Noting $u_i = u_e$ on Γ_a, we can see

$$-\int_\Omega u\Delta\overline{\psi}\,dx = \int_{\Omega_a} f\overline{\psi}\,dx - \langle \mathcal{T}u_i,\,\psi\rangle + \int_{\Omega_a'}\nabla u_e\cdot\nabla\overline{\psi}\,dx.$$

Thus, in order to show (70), it suffices to show that

$$(71)\quad \int_{\Omega_a'}\nabla u_e\cdot\nabla\overline{\psi}\,dx = \langle \mathcal{T}u_i,\,\psi\rangle.$$

We show (71) only in the two-dimensional case, since we can similarly show (71) in the three-dimensional case. Set $\varphi = u_i|_{\Gamma_a}$, and define, for $N \in \boldsymbol{N}$,

$$(72)\quad u_N = \sum_{n=-N}^{N}\left(\frac{a}{r}\right)^{|n|}\varphi_n Y_n(\theta) \qquad \text{on } \Omega_a'.$$

Then $u_N \in C^\infty(\overline{\Omega_a'})$, where $C^\infty(\overline{\Omega_a'}) = \{u = \tilde{u}|_{\Omega_a'} \mid \tilde{u} \in C^\infty(\boldsymbol{R}^d)\}$, and moreover $-\Delta u_N = 0$ in Ω_a'. Hence, by the Green formula, we obtain

$$(73)\quad \int_{\Omega_a'}\nabla u_N\cdot\nabla\overline{\psi}\,dx - \sum_{n=-N}^{N}\frac{|n|}{a}\varphi_n\int_{\Gamma_a}Y_n(\theta)\overline{\psi}\,d\gamma = 0 \qquad \text{for all } \psi \in C_0^\infty(\Omega).$$

There is a sufficiently large number b such that $\operatorname{supp}\psi \cap \Omega_a' \subset \Omega_a^b$, where $\Omega_a^b = \{x \in \boldsymbol{R}^d \mid a < |x| < b\}$. Hence we can rewrite (73) as follows:

$$(74)\quad \int_{\Omega_a^b}\nabla u_N\cdot\nabla\overline{\psi}\,dx = \sum_{n=-N}^{N}\frac{|n|}{a}\varphi_n\overline{\psi_n}.$$

Since $u_i|_{\Gamma_a} = \varphi \in H^{1/2}(\Gamma_a)$, we have

$$\sum_{n=-\infty}^{\infty}\frac{|n|}{a}\varphi_n\overline{\psi_n} = \langle \mathcal{T}u_i,\,\psi\rangle$$

and, by Lemma B.2,

$$u_N \longrightarrow u_e \qquad \text{in } H^1(\Omega_a^b).$$

Thus, forcing N to approach infinity in (74), we can get (71). ■

We finally have to prove Lemma B.2 to complete the proof of Theorem B.1.

Proof of Lemma B.2. We first prove the two-dimensional case. We can easily see that for all $n \in \boldsymbol{Z}$,

$$\left(\frac{a}{r}\right)^{|n|} Y_n(\theta) \in W(\Omega'_a).$$

For $N \in \boldsymbol{N}$, we define u_N by (72). Let us show that $u_N \longrightarrow u$ in $W(\Omega'_a)$ as $N \longrightarrow \infty$. For $N < N'$, we have

$$\begin{aligned}(75)\quad & \left\| \frac{u_N(x) - u_{N'}(x)}{(1+|x|^2)^{1/2}\log(2+|x|^2)} \right\|^2_{L^2(\Omega'_a)} \\ &= \sum_{N<|n|\le N'} \int_a^\infty \left(\frac{a}{r}\right)^{2|n|} \frac{r}{(1+r^2)\log^2(2+r^2)}\, dr a|\varphi_n|^2.\end{aligned}$$

Note that for all $n \in \boldsymbol{Z}$,

$$(76)\quad \int_a^\infty \left(\frac{a}{r}\right)^{2|n|} \frac{r}{(1+r^2)\log^2(2+r^2)}\, dr \le \int_a^\infty \frac{r}{(1+r^2)\log^2(2+r^2)}\, dr \equiv C_1 < +\infty.$$

From (75) and (76), we can get

$$(77)\quad \left\| \frac{u_N(x) - u_{N'}(x)}{(1+|x|^2)^{1/2}\log(2+|x|^2)} \right\|^2_{L^2(\Omega'_a)} \le aC_1 \sum_{N<|n|\le N'} |\varphi_n|^2.$$

We here note that we have

$$(78)\quad \int_{\Omega'_a} \nabla u \cdot \nabla v \, dx = \int_{\Omega'_a} \frac{\partial u}{\partial r}\frac{\partial v}{\partial r}\, dx - \int_{\Omega'_a} \frac{1}{r^2}(\Lambda u) v \, dx \qquad \text{for } u,\, v \in C^\infty(\overline{\Omega'_a}) \cap W(\Omega'_a),$$

where Λ is the Laplace-Beltrami operator on the unit circle of $\boldsymbol{R}^2$. Since $-\Lambda Y_n = n^2 Y_n$, we have, by (78),

$$\begin{aligned}(79)\quad & \|\nabla(u_N - u_{N'})\|^2_{L^2(\Omega'_a)} \\ &= \int_{\Omega'_a} \left| \frac{\partial u_N}{\partial r} - \frac{\partial u_{N'}}{\partial r} \right|^2 dx - \int_{\Omega'_a} \frac{1}{r^2} \left[\Lambda(u_N - u_{N'})\right] (\overline{u_N - u_{N'}})\, dx \\ &= \sum_{N<|n|\le N'} \int_a^\infty \left\{ \left| \frac{d}{dr}\left(\frac{a}{r}\right)^{|n|} \right|^2 + \frac{n^2}{r^2}\left(\frac{a}{r}\right)^{2|n|} \right\} r\, dr a|\varphi_n|^2 \\ &= \sum_{N<|n|\le N'} na|\varphi_n|^2.\end{aligned}$$

Since $\varphi \in H^{1/2}(\Gamma_a)$, it follows from (77) and (79) that $u_N \longrightarrow u$ in $W(\Omega'_a)$ as $N \longrightarrow \infty$.

We next prove the three-dimensional case. It is obvious that

$$\left(\frac{a}{r}\right)^{n+1} \varphi_n^m Y_n^m(\theta,\, \phi) \in W(\Omega'_a) \quad (n \in \boldsymbol{N} \cup \{0\},\ -n \le m \le n).$$

For $N \in \boldsymbol{N}$, we set

$$u_N = \sum_{n=0}^{N} \sum_{m=-n}^{n} \left(\frac{a}{r}\right)^{n+1} \varphi_n^m Y_n^m(\theta,\, \phi).$$

For $N < N'$, we have

$$(80)\quad \left\| \frac{u_N(x) - u_{N'}(x)}{(1+|x|^2)^{1/2}} \right\|^2_{L^2(\Omega'_a)} = \sum_{n=N+1}^{N'} \sum_{m=-n}^{n} \int_a^\infty \left(\frac{a}{r}\right)^{2(n+1)} \frac{r^2}{1+r^2}\, dr a^2 |\varphi_n^m|^2.$$

Here we note that for all $n \in \boldsymbol{N} \cup \{0\}$,

$$(81)\quad \int_a^\infty \left(\frac{a}{r}\right)^{2(n+1)} \frac{r^2}{1+r^2}\, dr \le \int_a^\infty \left(\frac{a}{r}\right)^2 dr \equiv C_2 < +\infty.$$

Form (80) and (81), we can get

$$(82)\quad \left\| \frac{u_N(x) - u_{N'}(x)}{(1+|x|^2)^{1/2}} \right\|^2_{L^2(\Omega'_a)} \le a^2 C_2 \sum_{n=N+1}^{N'} \sum_{m=-n}^{n} |\varphi_n^m|^2.$$

We here note that (78) holds good for the three-dimensional case. Therefore, since $-\Lambda Y_n^m = n(n+1) Y_n^m$, we have

$$\begin{aligned}
(83)\quad & \|\nabla(u_N - u_{N'})\|^2_{L^2(\Omega'_a)} \\
&= \int_{\Omega'_a} \left| \frac{\partial u_N}{\partial r} - \frac{\partial u_{N'}}{\partial r} \right|^2 dx - \int_{\Omega'_a} \frac{1}{r^2} \left[\Lambda(u_N - u_{N'})\right] \left(\overline{u_N - u_{N'}}\right) dx \\
&= \sum_{n=N+1}^{N'} \sum_{m=-n}^{n} \int_a^\infty \left\{ \left| \frac{d}{dr} \left(\frac{a}{r}\right)^{n+1} \right|^2 + \frac{n(n+1)}{r^2} \left(\frac{a}{r}\right)^{2(n+1)} \right\} r^2\, dr a^2 |\varphi_n^m|^2 \\
&= \sum_{n=N+1}^{N'} \sum_{m=-n}^{n} (n+1) a^3 |\varphi_n^m|^2.
\end{aligned}$$

Form (82) and (83), we can see that $u_N \longrightarrow u$ in $W(\Omega'_a)$ as $N \longrightarrow \infty$. ■

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References

[1] M. Abramowitz and I. A. Stegun: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. A Wiley-Interscience Publication, JOHN WILEY & SONS (1972).

[2] C. Amrouche, V. Girault, and J. Giroire: Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator: an approach in weighted Sobolev spaces. J. Math. Pures Appl. (9) **76**, no. 1, 55–81 (1997).

[3] M. O. Bristeau, R. Glowinski, and J. Périaux: Controllability methods for the computation of time-periodic solutions; application to scattering. J. Comput. Phys., **147**, no. 2, 265–292 (1998).

[4] B. Engquist and L. Halpern: Long-time behaviour of absorbing boundary conditions. Math. Methods Appl. Sci., **13**, no. 3, 189–203 (1990).

[5] M. J. Grote and J. B. Keller: On nonreflecting boundary conditions. J. Comput. Phys., **122**, no. 2, 231–243 (1995).

[6] M. Ikawa: Partial Differential Equations 2. Iwanami, Tokyo (1997) (in Japanese).

[7] D. Koyama: An artificial boundary condition for the wave equation and its application to a numerical algorithm for the Helmholtz equation (submitted).

[8] G. N. Watson: A Treatise on the Theory of Bessel Functions. Cambridge University Press (1922).