Appendix A

Equations of state

In this chapter we consider the relation between the Poisson relation in thermodynamics and the adiabatic equation in MHD [10, 12], which are rare to be described simultaneously in references. Moreover, we will show that the incompressibility condition can be derived as a singular limit of the adiabatic equation of state.

We define a fluid element as what occupies a local space surrounded by a certain boundary surfaces. In a plasma, the fluid element consists of innumerable charged particles, however, from macroscopic viewpoint, the fluid element must be sufficiently small in order to treat the macroscopic quantities as a common value. By assuming to take such an element for a plasma, we may define the plasma pressure, volume, and other macroscopic quantities for this element. Rigorously speaking, it is not evident whether the volume for fluid element is well defined. Since the density for the element can be defined, we define the volume in terms of local density of the fluid element. Therefore the volume is considered as a local quantity. Moreover, although the plasma consists of electrons and ions, we do not distinguish the difference and treat as a single fluid within the context of MHD.

Here we assume that each plasma element is at thermodynamical equilibrium in every time and every space. For p, V, n, R, T denoting pressure, volume, mol number of particles in an element, constant of gas, and temperature, respectively, Boyle-Charles' law;

$$pV = nRT \tag{A.1}$$

is assumed valid for a plasma.

For Q, S describing heat and entropy, the specific heat at constant volume C_v and that at constant pressure C_p are represented as

$$C_v = \left(\frac{\partial Q}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V, \tag{A.2}$$

$$C_p = \left(\frac{\partial Q}{\partial T}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p. \tag{A.3}$$

Here the subscripts for partial derivatives represent the fixed variable explicitly.

For $x = x(y, z)$, $y = y(z, x)$, $z = z(x, y)$ and $t = t(x, y, z)$, there are following relations;

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1, \tag{A.4}$$

$$\left(\frac{\partial x}{\partial y}\right)_z = \left(\frac{\partial x}{\partial y}\right)_t + \left(\frac{\partial x}{\partial t}\right)_y \left(\frac{\partial t}{\partial y}\right)_z. \tag{A.5}$$

At first, regarding x, y, z, t as S, T, V, p, respectively, in Eq. (A.5), and using equation of state (A.1) leads straightforwardly to the Mayer's relation;

$$C_p - C_v = nR. \tag{A.6}$$

And next, regarding x, y, z, t as p, V, S, T, respectively, Eq. (A.5) becomes

$$\left(\frac{\partial p}{\partial V}\right)_S = \left(\frac{\partial p}{\partial V}\right)_T + \left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_S. \tag{A.7}$$

Using Eq. (A.4), we can write the last term as

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial T}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T = -\frac{T}{C_v}\left(\frac{\partial p}{\partial T}\right)_V = -\frac{p}{C_v}. \tag{A.8}$$

In the last form of Eq. (A.8) we used the equation of state (A.1). Substituting this relation into Eq. (A.5) and using Mayer's relation (A.6), we obtain

$$\left(\frac{\partial p}{\partial V}\right)_S = -\frac{p}{V}\left(1 + \frac{nR}{C_v}\right) = -\frac{p}{V}\frac{C_p}{C_v}, \tag{A.9}$$

and integrating this equation leads to the following relation known as Poisson relation;

$$pV^\gamma = \text{const}, \tag{A.10}$$

where γ denotes the specific heat ratio $\gamma = C_p/C_v$. The relation (A.10) is called adiabatic law, since variation of the system with a fixed S means no heat conduction not only between the system and the outer region, but among all fluid elements. In other words, when the system suffers an adiabatic variation, the Poisson relation is applicable.

For a mass density of particles ρ, the volume is inversely proportional to the mass density, $V \propto \rho^{-1}$. Since the time derivative of Eq. (A.10) vanishes, the Poisson relation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p\rho^{-\gamma}\right) = 0, \tag{A.11}$$

where $\mathrm{d}/\mathrm{d}t = \partial_t + \boldsymbol{v} \cdot \nabla$ and $\boldsymbol{v}$ denotes a fluid velocity. Since we are looking at a certain fluid element, the time derivative should be taken in Lagrangian way. Substituting the continuity equation (2.1);

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho \nabla \cdot \boldsymbol{v}, \tag{A.12}$$

into Eq. (A.11), we obtain the pressure evolution equation (2.3);

$$\frac{\mathrm{d}p}{\mathrm{d}t} + \gamma p \nabla \cdot \boldsymbol{v} = 0. \tag{A.13}$$

For the degree of freedom of the macroscopic system N, the specific heat at constant volume is given as $C_v = NR/2$ from equipartition law of energy. Then the Mayer's relation (A.6) yields $\gamma = (N+2)/N$, i.e. for higher N, γ becomes smaller and approaches to unity. It can be interpreted intuitively that for the system with a larger degree of freedom, the fluid element can be deformed in a higher degree of freedom and can escape from the compression when external force is applied. Therefore, the adiabatic compression needs more pressure in the higher N than the lower one in order to decrease the same volume of the fluid element.

Following the above discussion, we can define the incompressibility condition as a singular limit of the adiabatic relation for the ideal gas (A.13). Incompressible fluid is that, the fluid element does not suffer any compression against any strong external pressure, i.e. n/V does not change for any large p. Even if the fluid element is deformed when exposed to an external force, however, it cannot be diverse freely and suffers the tightening from all direction due to its incompressibility, which is connected with the limit $N \to 0$. This gives a singular limit of the adiabatic equation of state $\gamma \to \infty$, which leads to the incompressible equation of state as

$$\nabla \cdot \boldsymbol{v} = 0. \tag{A.14}$$

By replacing the adiabatic equation of state (A.13) by the incompressible one (A.14), we can close the system of MHD equations.

In the same way, we can take the incompressible limit of the solution obtained by means of the compressible equation of state, however, it should be noted that the solution is consistent with the original adiabatic equation only in the limit $\gamma \to \infty$ in the way that

$$\gamma(\nabla \cdot v) \to -\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}. \tag{A.15}$$

This is the case discussed in Sec. 4.3. The limit $\gamma \to \infty$ corresponds to the situation where the sound wave will be excluded by putting $v_\mathrm{s} \to \infty$. Considering the fact that the phase velocity of the sound wave is much faster in the less compressible water than in the more compressible atmosphere, this result can be acceptable.

It is also noted that the condition of isothermal variation is obtained from Eq. (A.1) as $T = \text{const}$. Sometimes the anisotropic variation is considered. Such a case is discussed in Refs. [10, 12].