Appendix C

Electrostatic oscillations in an unmagnetized plasma

C.1 Langmuir oscillation

Let us first derive the dispersion relation of plasma oscillations by means of the fluid description. The governing equations for describing the one dimensional electrostatic oscillation (electron plasma oscillation) in an cold unmagnetized plasma are

$$\partial_t n + \partial_x (nv) = 0, \tag{C.1}$$

$$mn(\partial_t v + v\partial_x v) = qnE, \tag{C.2}$$

$$\epsilon_0 \partial_x E = qn. \tag{C.3}$$

Assuming the static homogeneous background plasma and linearizing Eqs. (C.1)-(C.3) for a plane wave with the dependence $e^{\mathrm{i}(kx-\omega t)}$ yields

$$-\mathrm{i}\omega n_1 + \mathrm{i}k n_0 v_1 = 0, \tag{C.4}$$

$$-\mathrm{i}\omega m v_1 = qE_1, \tag{C.5}$$

$$\mathrm{i}k\epsilon_0 E_1 = qn_1. \tag{C.6}$$

Combining Eqs. (C.5) and (C.6), we obtain

$$v_1 = \frac{q^2}{\epsilon_0 m\omega k} n_1. \tag{C.7}$$

Substituting Eq. (C.7) into Eq. (C.4) leads to the dispersion relation

$$\omega = \pm\omega_{\mathrm{p}} = \pm\sqrt{\frac{n_0 q^2}{\epsilon_0 m}}, \tag{C.8}$$

which is called a plasma oscillation.

It is known that the dispersion effect appears due to the finite electron pressure. Here we modify the equation of motion (C.2) as

$$mn(\partial_t v + v\partial_x v) = qnE - \partial_x p, \tag{C.9}$$

and add the adiabatic pressure equation

$$\partial_t p + v\partial_x p + \gamma p \partial_x v = 0, \tag{C.10}$$

as a closure of the fluid model. Linearizing these equations with the same plane wave dependence $e^{\mathrm{i}(kx-\omega t)}$, we have

$$-\mathrm{i}\omega m n_0 v_1 = q n_0 E_1 - \mathrm{i}k p_1, \tag{C.11}$$

$$-\mathrm{i}\omega p_1 + \mathrm{i}k\gamma p_0 v_1 = 0. \tag{C.12}$$

Substituting Eq. (C.12) into Eq. (C.11) and using Eq. (C.6) leads to

$$v_1 = \frac{\omega n_0 q^2}{k\epsilon_0(\omega^2 m n_0 - k^2\gamma p_0)} n_1. \tag{C.13}$$

Plugging Eq. (C.13) into (C.4) yields the dispersion relation

$$\begin{aligned} \omega^2 &= \omega_{\mathrm{p}}^2 + k^2 \frac{\gamma p_0}{m n_0}, \\ &= \omega_{\mathrm{p}}^2 + k^2 \frac{\gamma T_0}{m}, \end{aligned} \tag{C.14}$$

where we have used the fact that the electron pressure can be expressed by $p = nT$. Equation (C.14) explicitly shows the dispersion effect coming from the ∇p term in the equation of motion coupled with the adiabatic pressure equation.

C.2 Vlasov-Poisson system

The governing equations for describing Landau damping of one dimensional electrostatic oscillation (electron plasma oscillation) are

$$\partial_t f + v\partial_x f + \frac{qE}{m}\partial_v f = 0, \tag{C.15}$$

$$\epsilon_0 \partial_x E = q\int_{-\infty}^{+\infty} f \,\mathrm{d}v, \tag{C.16}$$

where f denotes the particle distribution function defined in the phase space, E the electric field, q and m the electric charge and mass of a particle (electron), ϵ_0 the vacuum susceptibility, respectively. Here x (v) denotes the coordinate space (velocity space) variable. Let us assume here that the background plasma is spatially homogeneous and electrically neutral ($E = 0$). For linearizing Eqs. (C.15) and (C.16), we can define a wave number k in the x direction. Combining these two equations with eliminating electric field yields

$$\partial_t f_1 + \mathrm{i}kv f_1 - \frac{\mathrm{i}q^2}{\epsilon_0 mk}(\partial_v f_0)\int_{-\infty}^{+\infty} f_1 \,\mathrm{d}v = 0, \tag{C.17}$$

where the subscripts 0 and 1 denote the equilibrium and the perturbation of distribution function, respectively. The distribution function f_1 belongs to $L^1(\mathbb{R})$ in the velocity space. Hereafter, we will omit the subscript 1 for simplicity.

Defining the Laplace transformation of the perturbed fields as

$$\tilde{\psi}(s) = \int_0^\infty \psi(t)\, e^{-st}\,\mathrm{d}t, \tag{C.18}$$

where the real part of s is chosen to be larger than any temporal singularity of the function $\psi(t)$ for the convergence of the integration. The inversion will be given by

$$\psi(t) = \frac{1}{2\pi\mathrm{i}}\int_{s_0-\mathrm{i}\infty}^{s_0+\mathrm{i}\infty} \tilde{\psi}(s)\, e^{st}\,\mathrm{d}s, \tag{C.19}$$

where $s_0 = \mathrm{Re}(s) > 0$.

Transforming Eq. (C.17) by multiplying e^{-st} and integrating with respect to time, we obtain

$$(s + \mathrm{i}kv)\tilde{f} - \frac{\mathrm{i}q^2}{\epsilon_0 mk}(\partial_v f_0)\int_{-\infty}^{+\infty} \tilde{f}\,\mathrm{d}v = f(v, t = 0). \tag{C.20}$$

Let us try an another calculation for comparison which can be seen in the literature [12]. Transforming Eqs. (C.15) and (C.16) by multiplying e^{-st} and integrating with respect to time yield

$$(s + \mathrm{i}kv)\tilde{f} + \frac{q}{m}\tilde{E}\partial_v f_0 = f(0), \tag{C.21}$$

$$\mathrm{i}k\epsilon_0\tilde{E} = q\int_{-\infty}^{+\infty} \tilde{f}\,\mathrm{d}v, \tag{C.22}$$

where the subscript 0 denotes the equilibrium field. The perturbations are Laplace transformed here. The initial condition for the electric field should be related with the distribution function through Poisson equation at $t = 0$ as

$$\mathrm{i}k\epsilon_0 E(0) = q\int_{-\infty}^{+\infty} f(v, 0)\,\mathrm{d}v. \tag{C.23}$$

Dividing Eq. (C.21) by $(s - \mathrm{i}kv)$ and plugging it into Eq. (C.22) lead to

$$\left[1 + \frac{q^2}{\epsilon_0 mk}\int_{-\infty}^{+\infty} \frac{\partial_v f_0}{\mathrm{i}s - kv}\,\mathrm{d}v\right]\tilde{E} = \frac{q}{\epsilon_0 k}\int_{-\infty}^{+\infty} \frac{f(0)}{\mathrm{i}s - kv}\,\mathrm{d}v. \tag{C.24}$$

The time evolution of the electric field can be obtained by inverting the Laplace transformation expressed in Eq. (C.19). From Eq. (C.24), we formally obtain

$$E(t) = \frac{1}{2\pi \mathrm{i}} \int_{s_0 - \mathrm{i}\infty}^{s_0 + \mathrm{i}\infty} \frac{\dfrac{q}{\epsilon_0 k} \displaystyle\int_{-\infty}^{+\infty} \frac{f(0)}{\mathrm{i}s - kv} \,\mathrm{d}v}{1 + \dfrac{q^2}{\epsilon_0 m k} \displaystyle\int_{-\infty}^{+\infty} \frac{\partial_v f_0}{\mathrm{i}s - kv} \,\mathrm{d}v} e^{st} \,\mathrm{d}s. \tag{C.25}$$

C.3 Spectrum of operators in Vlasov-Poisson system

The operator in the evolution equation (C.17) consists of two parts. In this subsection, we will discuss the properties of each part separately. The evolution equation for the perturbed distribution function is written as

$$\mathrm{i}\partial_t f = kvf - \frac{\omega_{\mathrm{p}}^2}{k} (\partial_v f_0) \int_{-\infty}^{+\infty} f \,\mathrm{d}v, \tag{C.26}$$

where we have normalized the equilibrium distribution function as

$$f_0(v) \to n_0 f_0(v). \tag{C.27}$$

C.3.1 Ballistic response

The first operator kv is the multiplication operator which gives rise to continuous spectrum on the whole real axis of λ, where λ is the spectrum of the operator defined by

$$\lambda f = kvf. \tag{C.28}$$

The spectra given by this multiplication operator are continuous ones and their generalized eigenfunctions are

$$f = \delta(v - \lambda/k), \tag{C.29}$$

where corresponding eigenvalues are

$$\lambda = kv. \tag{C.30}$$

The initial value problem of the multiplication operator is readily solved as

$$f(v,t) = e^{-\mathrm{i}kvt} f(v,0), \tag{C.31}$$

which is called the ballistic response of the plasma, since it describes the free streaming of particles with keeping their memory of initial disturbances [19]. The distribution function does not lose the initial memory $f(v,0)$, however, if we observe the integrated physical quantities such as density, there appears continuum damping given by

$$n(t) = \int_{-\infty}^{\infty} f(v,t)\,\mathrm{d}v \overset{t\to\infty}{\to} 0, \tag{C.32}$$

due to Riemann-Lebesgue theorem [28].

C.3.2 Operator $(\partial_v f_0)\int \cdot\, \mathrm{d}v$

Let us consider here the spectrum of the second operator $(\partial_v f_0)\int \cdot\, \mathrm{d}v$ in Eq. (C.26), which denotes the combination of linear functional and multiplication. Since the definite integral with respect to v and the multiplication of the function $(\partial_v f_0)(v)$ does not commute, this operator is non-Hermitian. The spectral problem is written as

$$\lambda f = -\frac{\omega_{\mathrm{p}}^2}{k} h(v) \int_{-\infty}^{\infty} f\,\mathrm{d}v, \tag{C.33}$$

where we have introduced $h(v) = \partial_v f_0$. Since the definite integral gives just a constant, it is clear that the eigenfunction is written as

$$f = ah(v), \tag{C.34}$$

where a is assumed as a constant coefficient. The eigenvalue is proportional to the integral of f. Suppose

$$\int_{-\infty}^{\infty} h(v)\,\mathrm{d}v = c \tag{C.35}$$

with a real number c, then we have the eigenvalue

$$\lambda = -\frac{\omega_{\mathrm{p}}^2}{k} ac. \tag{C.36}$$

For an equilibrium distribution function f_0 which gives a finite c (positive or negative), the spectrum of this operator continuously occupies the whole real axis, and all eigenfunctions are parallel and integrable. It is clear that such eigenfunctions will not span any physical linear space. However, to complete it is so difficult that we do not discuss how to solve this problem.

It seems strange that the eigenvalue depends on the magnitude of eigenfunction itself. In a realistic situation, however, we have to choose $f_0(v)$ as a member of a certain linear functional space, e.g. $L^1(\mathbb{R})$, thus

$$\int_{-\infty}^{\infty} h(v)\,\mathrm{d}v = \int_{-\infty}^{\infty} \partial_v f_0\,\mathrm{d}v \tag{C.37}$$

$$= [f_0(v)]_{-\infty}^{\infty} \tag{C.38}$$

holds. Consequently, the eigenvalue becomes zero if we take integrable $f_0(v)$.

C.4 Cold plasma with $f_0(v) = n_0\delta(v)$

Let us consider the cold electron plasma in this section by assuming

$$f_0(v) = n_0\delta(v). \tag{C.39}$$

In order to formulate the Hilbert space with following the discussion of Sec. 7.6, we have to include $\delta'(v)$ term in $f(v,t)$ as

$$f(v,t) = \alpha(t)\delta(v) + \beta(t)\delta'(v) \; + \; \varphi(v,t). \tag{C.40}$$

Here prime denotes the derivative with respect to its argument and $\varphi(v,t)$ denotes the continuous part of the perturbed distribution function, respectively. Then, the Laplace transformed equation (C.20) will give for the continuous part,

$$(s + \mathrm{i}kv)\tilde{\varphi}(v,s) = \varphi(v,0). \tag{C.41}$$

There appear couplings between singular surface wave parts. Using the formula $v\delta'(v) = -\delta(v)$, we obtain

$$s\tilde{\alpha}(s) - \mathrm{i}k\tilde{\beta}(s) = \alpha(0) \tag{C.42}$$

for the $\delta(v)$ component and

$$s\tilde{\beta}(s) - \frac{\mathrm{i}\omega_{\mathrm{p}}^2}{k}\tilde{\alpha}(s) - \frac{\mathrm{i}\omega_{\mathrm{p}}^2}{k}\int_{-\infty}^{\infty}\tilde{\varphi}(v,s)\,\mathrm{d}v = \beta(0) \tag{C.43}$$

for the $\delta'(v)$ component, respectively. Multiplying $(\mathrm{i}\omega_{\mathrm{p}}^2/k)$ on Eq. (C.42) and s on Eq. (C.43), and adding each other, we obtain

$$(s^2 + \omega_{\mathrm{p}}^2)\tilde{\beta}(s) - \frac{\mathrm{i}s\omega_{\mathrm{p}}^2}{k}\int_{-\infty}^{\infty}\tilde{\varphi}(v,s)\,\mathrm{d}v = \frac{\mathrm{i}\omega_{\mathrm{p}}^2}{k}\alpha(0) + s\beta(0). \tag{C.44}$$

From this equation,

$$\tilde{\beta}(s) = \frac{1}{(s - \mathrm{i}\omega_{\mathrm{p}})(s + \mathrm{i}\omega_{\mathrm{p}})}\left[\frac{\mathrm{i}\omega_{\mathrm{p}}^2}{k}\alpha(0) + s\beta(0) + \frac{\mathrm{i}s\omega_{\mathrm{p}}^2}{k}\int_{-\infty}^{\infty}\frac{\varphi(v,0)}{s + \mathrm{i}kv}\,\mathrm{d}v\right], \tag{C.45}$$

is given, where we have used Eq. (C.41). It is noted that this system also contains the resonance where the energy is transferred from the continuous spectrum to the point spectrum (surface wave). Inverting this expression, we obtain

$$\begin{aligned}\beta(t) = {} & \frac{\mathrm{i}\omega_{\mathrm{p}}}{k}\alpha(0)\sin(\omega_{\mathrm{p}}t) + \beta(0)\cos(\omega_{\mathrm{p}}t) \\ & + \frac{1}{2\pi\mathrm{i}}\int_{s_0-\mathrm{i}\infty}^{s_0+\mathrm{i}\infty}\frac{\mathrm{i}s\omega_{\mathrm{p}}^2/k}{(s - \mathrm{i}\omega_{\mathrm{p}})(s + \mathrm{i}\omega_{\mathrm{p}})}\int_{-\infty}^{\infty}\frac{\varphi(v,0)}{s + \mathrm{i}kv}\,\mathrm{d}v\,\mathrm{d}s.\end{aligned} \tag{C.46}$$

On the other hand, if we partially integrate the denominator of Eq. (C.25) as

$$\begin{aligned}\int \frac{\partial_v f_0}{\mathrm{i}s - kv}\,\mathrm{d}v &= \frac{n_0}{k}\int \frac{\delta'(v)}{v-(\mathrm{i}s/k)}\,\mathrm{d}v \\ &= \frac{n_0}{k}\left[\frac{\delta(v)}{v-(\mathrm{i}s/k)}\right]_{-\infty}^{+\infty} + \frac{n_0}{k}\int \frac{\delta(v)}{[v-(\mathrm{i}s/k)]^2}\,\mathrm{d}v \\ &= -\frac{n_0 k}{s^2},\end{aligned} \tag{C.47}$$

then the denominator of Eq. (C.25) will have zeros for

$$1 + \frac{q^2}{\epsilon_0 m k}\int_{-\infty}^{+\infty} \frac{\partial_v f_0}{\mathrm{i}s - kv}\,\mathrm{d}v = 0. \tag{C.48}$$

This gives

$$s_{\pm} = \pm \mathrm{i}\omega_{\mathrm{p}} \tag{C.49}$$

in the complex s-plane. In this case, we can formally rewrite $E(t)$ by substituting Eq. (C.47) into Eq. (C.25) as

$$E(t) = \frac{1}{2\pi}\int_{s_0-\mathrm{i}\infty}^{s_0+\mathrm{i}\infty} \frac{s^2 e^{st}}{(s-\mathrm{i}\omega_{\mathrm{p}})(s+\mathrm{i}\omega_{\mathrm{p}})}\frac{q}{\epsilon_0 k}\int_{-\infty}^{+\infty}\frac{f(0)}{s+\mathrm{i}kv}\,\mathrm{d}v\,\mathrm{d}s. \tag{C.50}$$

We consider the completely cold plasma by assuming

$$f(x,v,0) = \hat{n}_1 e^{\mathrm{i}kx}\delta(v), \tag{C.51}$$

where $\hat{n}_1$ denotes the real number expressing the amplitude of the initial disturbance. Then, the integration with respect to v is easily carried out,

$$E(t) = \frac{\hat{n}_1 e^{\mathrm{i}kx}}{2\pi}\frac{q}{\epsilon_0 k}\int_{s_0-\mathrm{i}\infty}^{s_0+\mathrm{i}\infty}\frac{s e^{st}}{(s-\mathrm{i}\omega_{\mathrm{p}})(s+\mathrm{i}\omega_{\mathrm{p}})}\,\mathrm{d}s. \tag{C.52}$$

This expression gives the simple oscillation

$$E(t) = \frac{\mathrm{i}q}{\epsilon_0 k}\hat{n}_1 e^{\mathrm{i}kx}\cos(\omega_{\mathrm{p}} t) \tag{C.53}$$

which exactly coincides with the analysis based on the fluid description.

On the other hand, if we introduce a finite temperature in the initial perturbation as

$$f(x,v,0) = \hat{n}_1 e^{\mathrm{i}kx} F(v), \tag{C.54}$$

where $F(v)$ denotes arbitrary analytic function. In this case, we can commute the integration with respect to v and s in Eqs. (C.46) and (C.50), and obtain the formal resonance which corresponds to the second order pole where

$$v = \pm\frac{\omega_{\mathrm{p}}}{k}, \tag{C.55}$$

is satisfied. By performing the integration with respect to s, we may be able to write the explicit form which only contains the integration with respect to v.

C.5 General continuous profile $f_0(v)$

In the case where $f_0(v)$ is a continuous function, van Kampen [90] and Case [49] have found the complete set of eigenfunctions. According to these references, all eigenvalues are real continuous ones in the case of Maxwellian equilibrium distribution, although the system may contain some complex point spectra in general. Construction of the propagator semi-group for the Vlasov-Poisson generator including such general distribution functions is discussed in Ref. [58]. Since the non-Hermitian operator $(\partial_v f_0) \int \cdot \, \mathrm{d}v$ gives fairly close effect to the inhomogeneous terms with its rank unity, they could have found those eigenfunctions by introducing the normalization of f in a tricky way. It is concluded that Landau's exponential damping for Maxwellian distribution function $f_0(v)$ does not denote a spectra of the operator, but just a consequence of the phase mixing damping due to the superposition of the continuous spectra.

It is pointed out by Weitzner [148, 149] that the Landau's prescription of taking a detour at the pole is not appropriate in general even though phase mixing will surely cause sometimes *non-exponential* Landau damping [in the sense of Eq. (C.32)]. However, it is experimentally confirmed to be *exponential* [102]. There might be something which we do not understand yet. It is also noted that a spatially inhomogeneous density profile will give rise to another continuous spectra in the coordinate space [39, 117].