Chapter 2

Single fluid magnetohydrodynamics

2.1 Magnetohydrodynamic equations

At first, we will introduce non-relativistic, single fluid, ideal magnetohydrodynamic (MHD) equations in SI units. Their derivation and validity are given in many books (see for example, Refs. [1, 3, 10, 19, 23]). Continuity equation and equation of motion are written as

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{2.1}$$

$$\rho(\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) = \boldsymbol{j} \times \boldsymbol{B} - \nabla p, \tag{2.2}$$

where ρ, $\boldsymbol{v}$, $\boldsymbol{j}$, $\boldsymbol{B}$ and p are fluid mass density, velocity, current density, magnetic field and pressure, respectively. Since the time scale of plasma dynamics is considerably faster than the heat conduction, we may assume that the each fluid element is insulated against heat exchange with its surroundings and locally in thermodynamic equilibrium. Consider the plasma as an ideal gas which follows the thermodynamical equation of state $p = nT$, where n and T denote the particle number density and the temperature in units of energy (J), respectively. Then the time evolution of pressure is shown as,

$$\partial_t p + \boldsymbol{v} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{v} = 0, \tag{2.3}$$

where γ denotes the specific heat ratio. It is noted that, when we do not treat the plasma as an ideal gas, e.g. incompressible fluid, we need another equation of state. These set of equations describe the dynamics of the plasma. We do not consider non-ideal kinetic effects including viscosity or resistivity of the plasma in this thesis.

For the magnetic field $\boldsymbol{B}$ and the electric field $\boldsymbol{E}$, we use the Maxwell equations.

One of them is Faraday's law,

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, \tag{2.4}$$

and another is Ampère's law,

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} \left(+ \frac{1}{c^2} \partial_t \boldsymbol{E} \right), \tag{2.5}$$

where μ_0 is the vacuum permeability, and c is the speed of the light, respectively. Here we note that Maxwell's displacement current (in the bracket) will often be neglected due to the smallness of its correction on MHD dynamics in non-relativistic regime. It is also related to the Galilei invariance of the equations, which we discuss in the next section. Since the plasma is assumed to be a perfectly conducting medium, the plasma resistivity is neglected and Ohm's law becomes

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \boldsymbol{0}, \tag{2.6}$$

which merely implies that the electric field will not appear in the rest frame of the plasma.

These equations consist a closed set of ideal MHD: i.e. for fourteen independent variables ρ, $\boldsymbol{v}$, $\boldsymbol{j}$, $\boldsymbol{B}$, p and $\boldsymbol{E}$, we have fourteen independent equations. In MHD equations, the Gauss' law for the electric field is not necessary since each fluid element is considered to be neutralized and charge separation is not treated. It is also noted that the Gauss' law for the magnetic field is considered as an initial condition. If it is initially satisfied, it will be kept forever as we can see by taking the divergence of Eq. (2.4)[1].

For later applications, let us further manipulate the above equations. Substituting Ohm's law (2.6) into Faraday's law (2.4) leads to the magnetic field induction equation

$$\partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = \boldsymbol{0}, \tag{2.7}$$

which enables us to eliminate the electric field from the governing equations. Moreover, by substituting Ampère's law (2.5) into the equation of motion (2.2), we can eliminate the plasma current from governing equations as

$$\rho(\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) = \frac{1}{\mu_0} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \nabla p, \tag{2.8}$$

where we have neglected the displacement current in Ampère's law.

[1] In two fluid theory, the former Gauss' law can be also understood as an initial condition. It is shown by taking the divergence of Eq. (2.5), denoting the electric charge as $\sigma = e(Zn_\mathrm{i} - n_\mathrm{e})$, and using continuity equations for electrons and ions.

In summary, the single fluid MHD equations are shown as

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{2.9}$$

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \frac{1}{\mu_0 \rho} \boldsymbol{B} \cdot \nabla \boldsymbol{B} - \frac{1}{\rho} \nabla \Big(p + \frac{B^2}{2\mu_0} \Big), \tag{2.10}$$

$$\partial_t p + \boldsymbol{v} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{v} = 0, \tag{2.11}$$

$$\partial_t \boldsymbol{B} + \boldsymbol{v} \cdot \nabla \boldsymbol{B} = -\boldsymbol{B}(\nabla \cdot \boldsymbol{v}) + \boldsymbol{B} \cdot \nabla \boldsymbol{v}. \tag{2.12}$$

There are eight independent variables ρ, $\boldsymbol{v}$, $\boldsymbol{B}$, and p, and corresponding eight independent evolution equations.

It is noted that the ideal MHD system has no characteristic scale length in general. They have two characteristic velocities, namely the Alfvén velocity and the sound velocity; however, there is no other typical scale. By taking any spatial scale with proportional to the time scale, we can write the equations into normalized form in any size. However, if we introduce a certain non-ideal effect, this property will be broken. For example, the Hall MHD system and the resistive MHD system contain the ion cyclotron frequency and the resistive diffusion time, respectively. They introduce the characteristic spatial scale when combined with the velocity one, namely the ion skin depth and the resistive skin depth.

2.2 Galilei invariance of Maxwell equations

Galilean transformation is defined as a small velocity limit of the Lorentz transformation [5]. It is clear that the non-relativistic fluid equations are Galilei invariant. However, the knowledge of relativity theory [25] is useful to show how the electromagnetic fields will be Galilean transformed. The Lorentz transformations of the electromagnetic fields in Gaussian units are given in Ref. [20]. Transformations of those expressions from Gaussian units to SI units can be done by means of the table in Ref. [17].

Let the inertial system K^* be moving with the relative velocity $\boldsymbol{V}$ with respect to the reference frame K. Then, the electromagnetic fields in the system K^* will be expressed in terms of that in the system K as

$$\boldsymbol{E}^*_\perp = \mathit{\Gamma}(\boldsymbol{E}_\perp + \boldsymbol{V} \times \boldsymbol{B}_\perp) \tag{2.13}$$

$$\boldsymbol{B}^*_\perp = \mathit{\Gamma}\Big(\boldsymbol{B}_\perp - \frac{1}{c^2} \boldsymbol{V} \times \boldsymbol{E}_\perp\Big), \tag{2.14}$$

where $\perp$ denotes the direction perpendicular to the relative velocity $\boldsymbol{V}$ between two systems. The parallel component of the field will not be changed. Here $\mathit{\Gamma} = (1 - V^2/c^2)^{-1/2}$ denotes the Lorentz factor. Charge density σ will be combined with the current density $\boldsymbol{j}$ to give a four-vector, therefore, they will be transformed as

$$\sigma^* = \Gamma\Big(\sigma - \frac{1}{c^2}\boldsymbol{j}\cdot\boldsymbol{V}\Big) \tag{2.15}$$

$$\boldsymbol{j}^* = \Gamma(\boldsymbol{j} - \sigma\boldsymbol{V}). \tag{2.16}$$

Of course, this combination of the transformation will not change Maxwell equations (including displacement current). With the relativistic equation of motion, they constitute the Lorentz invariant set of governing equations.

If we take the limit $|\boldsymbol{V}/c| \ll 1$, we obtain the following set of Galilean transformation relationships;

$$\boldsymbol{E}^* = \boldsymbol{E} + \boldsymbol{V}\times\boldsymbol{B}, \tag{2.17}$$

$$\boldsymbol{B}^* = \boldsymbol{B}, \tag{2.18}$$

$$\sigma^* = \sigma, \tag{2.19}$$

$$\boldsymbol{j}^* = \boldsymbol{j} - \sigma\boldsymbol{V}. \tag{2.20}$$

In the single fluid MHD equations, however, we have no charge separation which always give $\sigma = 0$ in the non-relativistic limit. Therefore, the current density will not be changed by the Galilean transformation. From these relations, it is readily shown that the pre-Maxwell equations (without displacement current) will not change their forms by the Galilean transformations (2.17)-(2.20). Thus, it is justified that the displacement current is neglected in the *non-relativistic* MHD model.

2.3 Conservation of energy

2.3.1 Nonlinear form

We will review the energy conservation relation and consider the effect of neglecting the displacement current again. Taking the scalar product of Eq. (2.2) with $\boldsymbol{v}$, the left hand side leads to

$$\begin{aligned}\rho\boldsymbol{v}\cdot(\partial_t\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}) &= \rho\partial_t\Big(\frac{1}{2}v^2\Big) + \rho\boldsymbol{v}\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{v}) + \frac{v^2}{2}[\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v})]\\ &= \partial_t\Big(\frac{1}{2}\rho v^2\Big) + \nabla\cdot\Big(\frac{1}{2}\rho v^2\boldsymbol{v}\Big).\end{aligned} \tag{2.21}$$

Here, in the first equality, we have added the left hand side of continuity equation Eq. (2.1) multiplied by $v^2/2$. In the second equality, we have used the vector relation $\boldsymbol{v}\cdot\nabla\boldsymbol{v} = \nabla(v^2/2) - \boldsymbol{v}\times(\nabla\times\boldsymbol{v})$. The second term of the right hand side of Eq. (2.2) will be evaluated as

$$\boldsymbol{v}\cdot\nabla p = \frac{1}{\gamma-1}\partial_t p + \frac{\gamma}{\gamma-1}\nabla\cdot(p\boldsymbol{v}), \tag{2.22}$$

with the adiabatic equation of state (2.3).

The first term of the right hand side of Eq. (2.2) will be evaluated by means of Maxwell equations. By taking scalar products of Eq. (2.4) with $\boldsymbol{B}/\mu_0$ and Eq. (2.5) with $\epsilon_0 \boldsymbol{E}$ and adding each other, we obtain

$$\partial_t \left[\left(\frac{\epsilon_0}{2} E^2 \right) + \frac{1}{2\mu_0} B^2 \right] = -\frac{1}{\mu_0} \nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) - \boldsymbol{j} \cdot \boldsymbol{E}, \tag{2.23}$$

where the first term of the right hand side denotes the Poynting vector, and the second term denotes Joule heat. The first term in the right hand side of equation of motion (2.2), therefore, will give

$$\begin{aligned} \boldsymbol{v} \cdot (\boldsymbol{j} \times \boldsymbol{B}) &= -\boldsymbol{j} \cdot (\boldsymbol{v} \times \boldsymbol{B}) = \boldsymbol{j} \cdot \boldsymbol{E} \\ &= -\partial_t \left[\left(\frac{\epsilon_0}{2} E^2 \right) + \frac{1}{2\mu_0} B^2 \right] - \frac{1}{\mu_0} \nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}), \end{aligned} \tag{2.24}$$

where we have used Ohm's law (2.6) in the second equality, and Eq. (2.23) in the last.

Adding up equalities (2.21), (2.22), and (2.24), we obtain the following local energy conservation relation:

$$\begin{aligned} &\partial_t \left[\frac{1}{2} \rho v^2 + \frac{1}{\gamma - 1} p + \left(\frac{\epsilon_0}{2} E^2 \right) + \frac{1}{2\mu_0} B^2 \right] \\ &\qquad = -\nabla \cdot \left[\frac{1}{2} \rho v^2 \boldsymbol{v} + \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B} + \frac{\gamma}{\gamma - 1} p \boldsymbol{v} \right], \end{aligned} \tag{2.25}$$

where the round bracket denotes the contribution of the displacement current. It is noted that the neglect of the displacement current leads to the exclusion of electric field energy from the conservation law. This may explain why the governing equations without displacement current are called 'magnetofluid' or 'magnetohydrodynamic' system of equations.

2.3.2 Linearized form and energy principle for static equilibria

Firstly, the physical quantities are divided into the equilibrium and the perturbation parts as

$$\psi = \psi_0 + \psi_1, \tag{2.26}$$

where subscripts 0 and 1 denote the equilibrium and perturbed quantities, respectively. Magnetohydrodynamic equilibria are defined by stationary class of solutions of the governing equations given by $\partial_t = 0$. These are expressed by the solutions of equations:

$$\nabla \cdot (\rho_0 \boldsymbol{v}_0) = 0, \tag{2.27}$$

$$\rho_0 \boldsymbol{v}_0 \cdot \nabla \boldsymbol{v}_0 = \boldsymbol{j}_0 \times \boldsymbol{B}_0 - \nabla p_0, \tag{2.28}$$

$$\boldsymbol{v}_0 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{v}_0 = \boldsymbol{0}, \tag{2.29}$$

$$\nabla \times (\boldsymbol{v}_0 \times \boldsymbol{B}_0) = \boldsymbol{0}, \tag{2.30}$$

$$\nabla \times \boldsymbol{B}_0 = \mu_0 \boldsymbol{j}_0. \tag{2.31}$$

It is noted that the magnetic field must also satisfy the divergence free condition ($\nabla \cdot \boldsymbol{B}_0 = 0$). In the case of static ($\boldsymbol{v}_0 = \boldsymbol{0}$) plasma, they can be very much simplified and give

$$\boldsymbol{j}_0 \times \boldsymbol{B}_0 = \nabla p_0, \tag{2.32}$$

which reduces to Grad-Shafranov equation in the toroidal axisymmetric case. General analyses of the equilibria with flows become a very profound problems even in the two dimensional case (see e.g. Ref. [31]); however, it is not the subject of this thesis. Later, we will discuss the linear spectral analyses for only simplified one dimensional model equilibria which satisfy the above equations almost trivially.

Suppose that such a static ($\boldsymbol{v}_0 = \boldsymbol{0}$) equilibrium is obtained, and let us introduce the displacement vector $\boldsymbol{\xi}$ for describing perturbations by

$$\partial_t \boldsymbol{\xi}(\boldsymbol{x}, t) = \boldsymbol{v}_1(\boldsymbol{x}, t), \quad \boldsymbol{\xi}(\boldsymbol{x}, 0) = \boldsymbol{0}. \tag{2.33}$$

Then, we can derive the evolution equation for $\boldsymbol{\xi}$ as

$$\begin{aligned} \partial_t^2 \boldsymbol{\xi} &= \mathcal{F}\boldsymbol{\xi} \\ &= \frac{1}{\rho_0}\Big[\nabla(\gamma p_0 \nabla \cdot \boldsymbol{\xi} + \boldsymbol{\xi} \cdot \nabla p_0) \\ &\quad + \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}_0) \times [\nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_0)] \\ &\quad + \frac{1}{\mu_0}[\nabla \times (\nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_0))] \times \boldsymbol{B}_0\Big]. \end{aligned} \tag{2.34}$$

After some tedious manipulations, it can be shown that the force operator $\mathcal{F}$ is Hermitian [10, 21] with respect to the scalar product

$$\langle \boldsymbol{\eta} \,|\, \boldsymbol{\xi} \rangle \equiv \frac{1}{2} \int_\Omega \rho_0 \bar{\boldsymbol{\eta}} \cdot \boldsymbol{\xi} \, \mathrm{d}V, \tag{2.35}$$

where the bar denotes the complex conjugate, and Ω denotes the plasma volume surrounded by a perfectly conducting wall. It is noted that this scalar product leads to the energy norm which plays a very important role in the later sections.

Hermiticity of the force operator $\mathcal{F}$ allows us to apply the spectral resolution due to von Neumann theorem [28]. Moreover, we can show the conservation of the perturbed energy

$$W = \frac{1}{2}\int_\Omega \rho_0 \left(|\partial_t \boldsymbol{\xi}|^2 - \bar{\boldsymbol{\xi}} \cdot \mathcal{F}\boldsymbol{\xi} \right) \mathrm{d}V, \tag{2.36}$$

by multiplying $\partial_t \bar{\boldsymbol{\xi}}$ on both sides of Eq. (2.34), $\partial_t \boldsymbol{\xi}$ on that of the complex conjugate of Eq. (2.34), and adding each side of equations. The conservation of W may also be shown with the triangular bracket defined by Eq. (2.35) as

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\langle \partial_t\boldsymbol{\xi} \,|\, \partial_t\boldsymbol{\xi}\rangle - \langle \boldsymbol{\xi} \,|\, \mathcal{F}\boldsymbol{\xi}\rangle\right) \\ &= \langle \partial_t^2\boldsymbol{\xi} \,|\, \partial_t\boldsymbol{\xi}\rangle + \langle \partial_t\boldsymbol{\xi} \,|\, \partial_t^2\boldsymbol{\xi}\rangle - \langle \partial_t\boldsymbol{\xi} \,|\, \mathcal{F}\boldsymbol{\xi}\rangle - \langle \boldsymbol{\xi} \,|\, \mathcal{F}\partial_t\boldsymbol{\xi}\rangle \\ &= \langle \partial_t^2\boldsymbol{\xi} - \mathcal{F}\boldsymbol{\xi} \,|\, \partial_t\boldsymbol{\xi}\rangle + \langle \partial_t\boldsymbol{\xi} \,|\, \partial_t^2\boldsymbol{\xi} - \mathcal{F}\boldsymbol{\xi}\rangle \\ &= 0,\end{aligned} \tag{2.37}$$

where we have used the Hermiticity of the force operator $\mathcal{F}$ in the third equality. Another important consequence of the Hermiticity is the energy principle [40, 96, 10, 21] which describes the necessary and sufficient condition for the MHD stability of static equilibria;

$$\delta W(\boldsymbol{\xi}, \boldsymbol{\xi}) \equiv -\langle \boldsymbol{\xi} \,|\, \mathcal{F}\boldsymbol{\xi}\rangle \geq 0 \quad (\text{for any } \boldsymbol{\xi}) \quad \longleftrightarrow \quad \text{stable}. \tag{2.38}$$

However, it is noted that the Hermiticity does not hold for shear flow plasmas as well as neutral fluids. Thus, these advantages for the linear stability theory will be lost in shear flow systems.

2.4 Magnetohydrodynamic waves in homogeneous plasmas

Let us review the small amplitude waves in the ideal MHD system for homogeneous plasmas. If the plasma is flowing with a homogeneous velocity, it generates just a uniform Doppler shift of wave frequencies. Thus, we consider static plasma here without loss of generality. Consider the equilibrium magnetic field to be in the z direction of the Cartesian coordinates as

$$\boldsymbol{B}_0 = (0, 0, B_{0z}). \tag{2.39}$$

Physical quantities are linearized with respect to perturbations as

$$\psi = \psi_0 + \psi_1, \tag{2.40}$$

where subscripts 0 and 1 denote the equilibrium and perturbed quantities, respectively. Assume $e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$ dependence for perturbed fields, where $\boldsymbol{k} = (0, k_\perp, k_\parallel)$ in the Cartesian coordinates. Then, Eqs. (2.9)-(2.12) will be combined and written after linearization as

$$-\mathrm{i}\omega \begin{pmatrix} \rho \\ p \\ v_x \\ v_y \\ v_z \\ B_x \\ B_y \\ B_z \end{pmatrix} = \mathcal{A} \begin{pmatrix} \rho \\ p \\ v_x \\ v_y \\ v_z \\ B_x \\ B_y \\ B_z \end{pmatrix}, \tag{2.41}$$

where the matrix $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i}k_\perp\rho_0 & \mathrm{i}k_\parallel\rho_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i}k_\perp\gamma p_0 & -\mathrm{i}k_\parallel\gamma p_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{i}k_\parallel B_{0z}}{\mu_0\rho_0} & 0 & 0 \\ 0 & -\frac{\mathrm{i}k_\perp}{\rho_0} & 0 & 0 & 0 & 0 & \frac{\mathrm{i}k_\parallel B_{0z}}{\mu_0\rho_0} & -\frac{\mathrm{i}k_\perp B_{0z}}{\mu_0\rho_0} \\ 0 & -\frac{\mathrm{i}k_\parallel}{\rho_0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i}k_\parallel B_{0z} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i}k_\parallel B_{0z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i}k_\perp B_{0z} & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.42}$$

Here, we have omitted the subscript 1 denoting perturbed quantities for simplicity. The matrix $\mathcal{A}$ may not seem to be anti-Hermitian due to the asymmetry of the components $\mathcal{A}_{14,15}$ and $\mathcal{A}_{41,51}$, however, it can be removable due to the vacancy of the first column. It is also noted that the first and the second row is parallel with each other. Physically it means that the time evolution equations which show the continuity and the adiabatic state are not purely independent. One of the two equations (and thus, one of two physical quantities, namely density or pressure) can be removable from the system by means of the relation

$$p = c_\mathrm{s}^2\rho^\gamma, \tag{2.43}$$

where c_s denotes the local phase velocity of the sound wave. After the reduction of the physical quantities, we can transform the matrix $\mathcal{A}$ into anti-Hermitian one by introducing the appropriate normalizations. Reduced 7×7 anti-Hermitian matrix contains the seven orthogonal eigenvectors and the corresponding eigenvalues. All eigenvalues of the anti-Hermitian matrix give pure imaginary numbers, namely real ω's. Thus, there are seven types of small amplitude waves in the ideal MHD system.

Alfvén waves Since the v_x and B_x components can be easily decoupled from others in Eq. (2.41), we can readily obtain the Alfvén wave dispersion relation as

$$\omega = \pm k_{\parallel} v_{\mathrm{A}}, \tag{2.44}$$

where the corresponding eigenvectors are written as

$$(v_x, B_x) = (\mp 1, \sqrt{\mu_0 \rho_0}), \quad \rho = p = v_y = v_z = B_y = B_z = 0. \tag{2.45}$$

Here the phase velocity of the Alfvén wave is introduced as $v_{\mathrm{A}} = B_{0z}/\sqrt{\mu_0 \rho_0}$. It is noted that, since $\boldsymbol{k} \cdot \boldsymbol{v} = 0$, Alfvén wave shows an incompressible transverse perturbation of the plasma element.

Entropy wave and Magnetosonic waves The remaining five waves satisfy the following eigenvalue problem:

$$-\mathrm{i}\omega \begin{pmatrix} p \\ v_y \\ v_z \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{i}k_{\perp}\gamma p_0 & -\mathrm{i}k_{\parallel}\gamma p_0 & 0 & 0 \\ -\frac{\mathrm{i}k_{\perp}}{\rho_0} & 0 & 0 & \frac{\mathrm{i}k_{\parallel}B_{0z}}{\mu_0\rho_0} & -\frac{\mathrm{i}k_{\perp}B_{0z}}{\mu_0\rho_0} \\ -\frac{\mathrm{i}k_{\parallel}}{\rho_0} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{i}k_{\parallel}B_{0z} & 0 & 0 & 0 \\ 0 & -\mathrm{i}k_{\perp}B_{0z} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ v_y \\ v_z \\ B_y \\ B_z \end{pmatrix}. \tag{2.46}$$

The eigenvalues are obtained by means of the sweeping-out method, which leads to the dispersion relation

$$\omega[\omega^4 - k^2(v_{\mathrm{A}}^2 + v_{\mathrm{s}}^2)\omega^2 + k^2 k_{\parallel}^2 v_{\mathrm{A}}^2 v_{\mathrm{s}}^2] = 0, \tag{2.47}$$

where $v_{\mathrm{s}} = \sqrt{\gamma p_0/\rho_0}$ denotes the phase velocity of the sound wave. Thus, it is found that there are an entropy wave which satisfies

$$\omega = 0, \tag{2.48}$$

and magnetosonic waves which satisfy

$$\omega^2 = \frac{1}{2}\left[k^2(v_{\mathrm{A}}^2 + v_{\mathrm{s}}^2) \pm \sqrt{k^4(v_{\mathrm{A}}^2 + v_{\mathrm{s}}^2)^2 - 4k^2 k_{\parallel}^2 v_{\mathrm{A}}^2 v_{\mathrm{s}}^2}\right]. \tag{2.49}$$

Here, two of the magnetosonic waves (+ sign) denote fast waves and the other two (− sign) denote slow waves, respectively, and $k^2 = k_{\perp}^2 + k_{\parallel}^2$.

In the case of $k_{\parallel} = 0$ in Eq. (2.47), we obtain the following eigenvalues

$$\omega = 0, \tag{2.50}$$

$$\omega = 0, \tag{2.51}$$

$$\omega = \pm k_{\perp}\sqrt{v_{\mathrm{A}}^2 + v_{\mathrm{s}}^2}, \tag{2.52}$$

and the corresponding eigenvectors

$$\rho \neq 0,\ p \neq 0,\ B_z \neq 0, \quad v_x = v_y = v_z = B_x = B_y = 0, \tag{2.53}$$

$$v_z \neq 0, B_y \neq 0, \quad \rho = p = v_x = v_y = B_x = B_z = 0, \tag{2.54}$$

$$\rho \neq 0,\ p \neq 0,\ v_y \neq 0,\ B_z \neq 0, \quad v_x = v_z = B_x = B_y = 0, \tag{2.55}$$

respectively. It is noted that the slow waves are degenerated here to give zero eigenvalues. The fast waves are reduced to a couple of oppositely propagating compressional Alfvén waves.

By putting $k_\perp = 0$ in Eq. (2.47), we can decouple two magnetosonic waves and the eigenvalues become

$$\omega = 0, \tag{2.56}$$

$$\omega = \pm k_\parallel v_\mathrm{s}, \tag{2.57}$$

$$\omega = \pm k_\parallel v_\mathrm{A}, \tag{2.58}$$

where the corresponding eigenvectors are shown as

$$B_z \neq 0, \quad \rho = p = v_x = v_y = v_z = B_x = B_y = 0, \tag{2.59}$$

$$\rho \neq 0,\ p \neq 0,\ v_z \neq 0, \quad v_x = v_y = B_x = B_y = B_z = 0, \tag{2.60}$$

$$v_y \neq 0,\ B_y \neq 0, \quad \rho = p = v_x = v_z = B_x = B_z = 0, \tag{2.61}$$

respectively. In this case, the slow waves reduce to the sound waves which propagate with the same mechanism in neutral fluids. It is noted that the sound waves do not carry any electric field ($E_\parallel = 0$) in the description of single fluid MHD equations.[2] The fast waves reduce to the degenerated shear Alfvén waves whose eigenvalues and eigenvectors are same as Eqs. (2.44) and (2.45), since x and y directions are not distinguishable in this situation.

In general, the magnetosonic wave accompanies compression of the plasma, $\boldsymbol{k} \cdot \boldsymbol{v} \neq 0$. This means that these two branches will be excluded from the system by assuming incompressibility on the perturbed velocity. However, as will be discussed in Sec. 4.3, we should be careful for an additional condition on the original MHD system. Actually, incompressibility is consistent with the adiabatic pressure equation only in the limit $\gamma \to \infty$ in the way that

$$\gamma(\nabla \cdot v) \to -\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}. \tag{2.62}$$

The limit $\gamma \to \infty$ corresponds to the situation that the sound wave will be excluded from the system with $v_\mathrm{s} \to \infty$. For the mathematical discussions, see Ref. [116] and the references therein. The problem encountered in the determination of the pressure for the incompressible fluid is also discussed in Ref. [93].

[2] On the contrary, in the two fluid theory, electric field plays a role in the propagation of ion sound waves.

2.5 Linearized one dimensional reduced magnetohydrodynamic equations

In this section, we will consider the linearized one dimensional reduced magnetohydrodynamic (RMHD) equations for a low beta static plasma. The first simplification was done by Strauss [123].

2.5.1 Derivation

Here we will simplify the derivation without discussing the detailed physical situations. It is noted that the MHD equilibria are described as

$$\boldsymbol{j}_0 \times \boldsymbol{B}_0 = \nabla p_0, \tag{2.63}$$

where the plasma current density is taken $\boldsymbol{j}_0 = j_0 \boldsymbol{e}_z$ with $\boldsymbol{e}_z$ denoting the unit vector in the z direction, and the strong magnetic field is applied in the z direction.

Under the above situation, we may assume that the perturbation fields come from two dimensional incompressible motions and are written as

$$\boldsymbol{v}_1 = \nabla\phi \times \boldsymbol{e}_z, \quad \boldsymbol{B}_1 = \nabla\psi \times \boldsymbol{e}_z. \tag{2.64}$$

Here, ϕ and ψ denote the stream function and the flux function, respectively. By assuming $\rho = \rho_0 =$ const, the continuity equation becomes a trivial relation. It is noted that the incompressibility is a valuable relation to assume for the simple description of (magneto)fluids, however, such a simplification sometimes spoils physical consistency. Therefore, we have to be careful for introducing additional constraints. The consistency of the incompressibility assumption is discussed in the Appendix A and Sec. 4.3.

Taking the curl of the equation of motion (2.10) gives the vorticity evolution equation;

$$\rho_0 \partial_t (\nabla \times \boldsymbol{v}_1) = \boldsymbol{B}_0 \cdot \nabla \boldsymbol{j}_1 + \boldsymbol{B}_1 \cdot \nabla \boldsymbol{j}_0 - \boldsymbol{j}_0 \cdot \nabla \boldsymbol{B}_1 - \boldsymbol{j}_1 \cdot \nabla \boldsymbol{B}_0. \tag{2.65}$$

Here, the third and the last terms in the right hand side can be omitted when we take the z component of the vorticity equation. It is because the spatial variation of the z component in equilibrium and perturbed magnetic fields are negligible under the application of strong magnetic field B_{0z}. By substituting Eq. (2.64), we can rewrite Eq. (2.65) in terms of stream function ϕ and flux function ψ as

$$\partial_t \Delta\phi = \frac{1}{\mu_0 \rho_0} \boldsymbol{B}_0 \cdot \nabla \Delta\psi + \frac{1}{\rho_0} (\nabla j_0 \times \boldsymbol{e}_z) \cdot \nabla\psi, \tag{2.66}$$

where $\Delta = \partial_x^2 + \partial_y^2$ denotes the two dimensional Laplacian operator in the perpendicular direction of the ambient strong magnetic field (z direction).

Let us now formulate the induction equation. It is noted that the right hand side of the induction equation (2.7) can be manipulated to give

$$\begin{aligned} \nabla \times (\boldsymbol{v}_1 \times \boldsymbol{B}_0) &= \nabla \times [(\nabla\phi \times \boldsymbol{e}_z) \times \boldsymbol{B}_0] \\ &= \nabla \times [(\boldsymbol{B}_0 \cdot \nabla\phi)\boldsymbol{e}_z], \end{aligned} \tag{2.67}$$

where we have assumed that the axial equilibrium magnetic field is homogeneous. Moreover, by using the relation $\nabla\psi \times \boldsymbol{e}_z = \nabla \times (\psi \boldsymbol{e}_z)$, Eq. (2.7) leads to

$$\partial_t[\nabla \times (\psi \boldsymbol{e}_z)] = \nabla \times [(\boldsymbol{B}_0 \cdot \nabla\phi)\boldsymbol{e}_z]. \tag{2.68}$$

If we omit the curl operator on both sides of this equation, we obtain

$$\partial_t \psi = \boldsymbol{B}_0 \cdot \nabla\phi, \tag{2.69}$$

where it is shown in Ref. [123] that the arbitrariness of the gradient field may be neglected due to the ambient strong magnetic field B_{0z}.

The somewhat different formalisms which lead to three fields evolution equations are seen for high beta tokamaks [124] or stellarators [125, 27, 32]. The latter will be used in the analysis in Chap. 3.

2.5.2 Hermiticity of Alfvén operator

Here we discuss the formal Hermiticity of the Alfvén operator embedded in the energy norm. First we introduce the matrix representation of the linearized RMHD equations (2.66) and (2.69). Let B, a, and $\tau_{\mathrm{A}} = a\sqrt{\mu_0\rho_0}/B$ be the characteristic magnetic field strength, scale length, and time scale, respectively. Then the physical quantities are normalized as

$$\phi \to \frac{a^2}{\tau_{\mathrm{A}}}\phi, \quad \psi \to aB\psi, \quad \boldsymbol{B}_0 \to B\boldsymbol{B}, \quad j_0 \to \frac{B}{a\mu_0}j. \tag{2.70}$$

With a state vector $u = {}^{T}(\Delta\phi, \psi)$, the evolution equations (2.66) and (2.69) are combined and written in the operator matrix form as

$$\partial_t u = \mathcal{A}u, \tag{2.71}$$

where the operator matrix $\mathcal{A}$ is defined as

$$\mathcal{A} = \begin{pmatrix} 0 & \boldsymbol{B} \cdot \nabla\Delta + \nabla j \times \boldsymbol{e}_z \cdot \nabla \\ \boldsymbol{B} \cdot \nabla\Delta^{-1} & 0 \end{pmatrix}, \tag{2.72}$$

and the superscript T denotes the transpose of the matrix.

Magnetic derivative operator Let ϕ and ϕ^* be two scalar functions defined in the plasma domain Ω satisfying the boundary condition

$$\phi = 0, \quad \phi^* = 0 \quad \text{on } \partial\Omega. \tag{2.73}$$

Then, it is readily shown that the magnetic derivative operator $\boldsymbol{B} \cdot \nabla$ is anti-Hermitian with the simple norm

$$(\phi \,|\, \phi^*) = \int \bar{\phi}\phi^* \, \mathrm{d}V \tag{2.74}$$

according to the equality

$$\begin{aligned}(\bar{\phi}\boldsymbol{B}) \cdot \nabla\phi^* &= \nabla \cdot (\bar{\phi}\phi^*\boldsymbol{B}) - [\nabla \cdot (\bar{\phi}\boldsymbol{B})]\phi^* \\ &= \nabla \cdot (\bar{\phi}\phi^*\boldsymbol{B}) - (\boldsymbol{B} \cdot \nabla\bar{\phi})\phi^*,\end{aligned}$$

which comes from Gauss' law $\nabla \cdot \boldsymbol{B} = 0$. It reads as

$$(\phi \,|\, \boldsymbol{B} \cdot \nabla\phi^*) = -(\boldsymbol{B} \cdot \nabla\phi \,|\, \phi^*). \tag{2.75}$$

Norm of state vector u It is clear that Hermiticity condition is not obtained with the simple norm defined by Eq. (2.74) for the state vector u. We will introduce a 'modified norm' here. Let $u = {}^T(\Delta\phi, \psi)$ and $u^* = {}^T(\Delta\phi^*, \psi^*)$ be two state vectors. By taking the metric as

$$\mathcal{M} = \begin{pmatrix} -\Delta^{-1} & 0 \\ 0 & -\Delta \end{pmatrix}, \tag{2.76}$$

we can define the formal scalar product as

$$\langle u \,|\, u^* \rangle \equiv (\Delta\phi \,|\, -\Delta^{-1} \,|\, \Delta\phi^*) + (\psi \,|\, -\Delta \,|\, \psi^*), \tag{2.77}$$

where $(\ |\ )$ denotes the simple norm defined by Eq. (2.74). Physically, it is shown that this metric gives the bilinear form corresponding to perturbed energy as

$$\langle u \,|\, u \rangle = \int \Delta\bar{\phi}(-\Delta^{-1})\Delta\phi + \bar{\psi}(-\Delta)\psi \, \mathrm{d}V \tag{2.78}$$

$$= \int |\nabla\phi|^2 + |\nabla\psi|^2 \, \mathrm{d}V, \tag{2.79}$$

where we have omitted the factor 1/2 for simplicity.

Anti-Hermiticity of $\mathcal{A}$ with homogeneous $\boldsymbol{B}$ Firstly, we will assume here that the magnetic field is spatially homogeneous. In this case, the current density j is eliminated and $\mathcal{A}$ becomes

$$\mathcal{A}_\mathrm{h} = \begin{pmatrix} 0 & \boldsymbol{B} \cdot \nabla\Delta \\ \boldsymbol{B} \cdot \nabla\Delta^{-1} & 0 \end{pmatrix}. \tag{2.80}$$

By taking two state vectors u and u^* as before, we can show the anti-Hermiticity in the matrix form after careful calculations;

$$\begin{aligned}
\langle u \,|\, \mathcal{A}_{\mathrm{h}} u^* \rangle &\equiv \int (\Delta\bar{\phi}, \bar{\psi})\, \mathcal{M}\mathcal{A}_{\mathrm{h}} \begin{pmatrix} \Delta\phi^* \\ \psi^* \end{pmatrix} \mathrm{d}V \\
&= -\int (\Delta\bar{\phi}, \bar{\psi}) \begin{pmatrix} 0 & \Delta^{-1}\boldsymbol{B}\cdot\nabla\Delta \\ \Delta\boldsymbol{B}\cdot\nabla\Delta^{-1} & 0 \end{pmatrix} \begin{pmatrix} \Delta\phi^* \\ \psi^* \end{pmatrix} \mathrm{d}V \\
&= -\int (\Delta\bar{\phi})(\Delta^{-1}\boldsymbol{B}\cdot\nabla\Delta\psi^*) + \bar{\psi}(\Delta\boldsymbol{B}\cdot\nabla\phi^*)\,\mathrm{d}V \\
&= -\int (\boldsymbol{B}\cdot\nabla\Delta^{-1}\Delta\bar{\phi})(-\Delta)(\psi^*) + (\boldsymbol{B}\cdot\nabla\Delta\bar{\psi})(-\Delta^{-1})(\Delta\phi^*)\,\mathrm{d}V \\
&= \int^T \left\{ \begin{pmatrix} 0 & -\boldsymbol{B}\cdot\nabla\Delta \\ -\boldsymbol{B}\cdot\nabla\Delta^{-1} & 0 \end{pmatrix} \begin{pmatrix} \Delta\bar{\phi} \\ \bar{\psi} \end{pmatrix} \right\} \\
&\quad \times \begin{pmatrix} -\Delta^{-1} & 0 \\ 0 & -\Delta \end{pmatrix} \begin{pmatrix} \Delta\phi^* \\ \psi^* \end{pmatrix} \mathrm{d}V \\
&= -\langle \mathcal{A}_{\mathrm{h}} u \,|\, u^* \rangle, \qquad (2.81)
\end{aligned}$$

where we have used the Hermiticity and anti-Hermiticity of the operator Δ and $\boldsymbol{B}\cdot\nabla$ with the simple norm, respectively. Since all eigenvalues of the anti-Hermitian operator are pure imaginary, the time evolution of $\mathcal{A}$ will give the simple oscillatory behavior representing the Alfvén wave. However, it is difficult to check the Hermiticity for the operator matrix form in the case of inhomogeneous magnetic field. The reason is the existence of kink instability.

Hermiticity of unified scalar Alfvén operator Let us consider then the unified scalar Alfvén operator. Combining Eqs. (2.66) and (2.69), we can write the unified equation for the vorticity $\Delta\phi$ as

$$\partial_t^2 \Delta\phi = \boldsymbol{B}\cdot\nabla\Delta\boldsymbol{B}\cdot\nabla\Delta^{-1}(\Delta\phi) + (\nabla j \times \boldsymbol{e}_z)\cdot\nabla\boldsymbol{B}\cdot\nabla\Delta^{-1}(\Delta\phi). \qquad (2.82)$$

If we just consider the operator

$$\boldsymbol{B}\cdot\nabla\Delta\boldsymbol{B}\cdot\nabla + (\nabla j \times \boldsymbol{e}_z)\cdot\nabla\boldsymbol{B}\cdot\nabla \qquad (2.83)$$

for the stream function ϕ with the simple norm (2.74), it seems Hermitian because the second term yields a multiplication operator for the one dimensional MHD equilibrium. However, since Eq. (2.82) is an evolution equation for the vorticity, we should consider the following generator

$$\mathcal{A}_{\mathrm{u}} = \boldsymbol{B}\cdot\nabla\Delta\boldsymbol{B}\cdot\nabla\Delta^{-1} + (\nabla j \times \boldsymbol{e}_z)\cdot\nabla\boldsymbol{B}\cdot\nabla\Delta^{-1}, \qquad (2.84)$$

for the vorticity, and we should take the energy norm as discussed in the previous paragraph. We just consider the one dimensional MHD equilibrium and take $\boldsymbol{B}$ in the yz (θz) plane depending only on x (r) in the Cartesian (cylindrical) coordinates. Then, the operator $(\nabla j \times \boldsymbol{e}_z) \cdot \nabla \boldsymbol{B} \cdot \nabla$ reduces to a multiplication operator with wave numbers in y and z (θ and z) directions. By introducing

$$f(\boldsymbol{x}) = (\nabla j \times \boldsymbol{e}_z) \cdot \nabla \boldsymbol{B} \cdot \nabla. \tag{2.85}$$

we may simplify the expression of the generator as

$$\mathcal{A}_\mathrm{u} = \boldsymbol{B} \cdot \nabla \Delta \boldsymbol{B} \cdot \nabla \Delta^{-1} + f(\boldsymbol{x}) \Delta^{-1}. \tag{2.86}$$

With the energy norm by following Eq. (2.77) as

$$\langle \Delta\phi \,|\, \Delta\phi^* \rangle = (\Delta\phi \,|\, -\Delta^{-1} \,|\, \Delta\phi^*), \tag{2.87}$$

where (|) denoting the simple norm (2.74), the Hermiticity of the operator $\mathcal{A}_\mathrm{u}$ is shown as

$$\begin{aligned}
\langle \Delta\phi \,|\, \mathcal{A}_\mathrm{u} \Delta\phi^* \rangle &= -(\Delta\phi \,|\, \Delta^{-1} \boldsymbol{B} \cdot \nabla \Delta \boldsymbol{B} \cdot \nabla \Delta^{-1} \Delta\phi^*) - (\Delta\phi \,|\, \Delta^{-1} f(\boldsymbol{x}) \Delta^{-1} \Delta\phi^*) \\
&= (\boldsymbol{B} \cdot \nabla \Delta \boldsymbol{B} \cdot \nabla \Delta^{-1} \Delta\phi \,|\, -\Delta^{-1} \,|\, \Delta\phi^*) \\
&\quad + (f(\boldsymbol{x}) \Delta^{-1} \Delta\phi \,|\, -\Delta^{-1} \,|\, \Delta\phi^*) \\
&= \langle \mathcal{A}_\mathrm{u} \Delta\phi \,|\, \Delta\phi^* \rangle.
\end{aligned} \tag{2.88}$$

Here we have used the Hermiticity and anti-Hermiticity of the operators Δ and $\boldsymbol{B} \cdot \nabla$ with the simple norm, respectively.

2.6 Spectra of Alfvén waves in static equilibria

A complete spectral ordinary differential equation for studying MHD perturbations in a static cylindrical plasma (general screw pinch) is derived by Hain and Lüst [76]. Instead of Hain-Lüst equation, we will treat a simpler equation under the assumption of incompressibility. The spectral properties of the Alfvén wave are focused on in the slab geometry (Sec. 2.6.1) and in the cylindrical geometry (Sec. 2.6.2). We will start from the unified scalar Alfvén equation (2.82) without normalization;

$$\partial_t^2 \Delta\phi = \frac{1}{\mu_0 \rho} \boldsymbol{B} \cdot \nabla \Delta \boldsymbol{B} \cdot \nabla \phi + \frac{1}{\rho} (\nabla j \times \boldsymbol{e}_z) \cdot \nabla \boldsymbol{B} \cdot \nabla \phi, \tag{2.89}$$

where we have omitted the subscript 0 denoting the equilibrium quantities. The assumptions which we have imposed here are the *incompressibility*

$$\nabla \cdot \boldsymbol{v} = 0 \tag{2.90}$$

of the ideal MHD plasma instead of the adiabatic equation of state (2.3), and the *one dimensionality* of the *static* equilibrium. The variation of the equilibrium quantities are taken in the x (r) direction of the Cartesian (cylindrical) coordinate system.

2.6.1 Slab geometry — continuous spectra —

The equilibrium magnetic field is assumed as

$$\boldsymbol{B} = (0, B_y(x), B_z), \tag{2.91}$$

with $B_z = \text{const}$ in the Cartesian coordinates. From the homogeneity of the equilibrium quantities in the y and z directions, the wave numbers in both directions become good quantum numbers and we take $\boldsymbol{k} = (0, k_y, k_z)$. Since the generator is Hermitian as shown in Sec. 2.5.2, we may consider the eigenvalue λ of the generator with replacing ∂_t by $-\mathrm{i}\omega$ ($\lambda = -\omega^2$). Then, Eq. (2.89) gives

$$-\omega^2\Big(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - k_y^2\Big)\phi = -\frac{F}{\mu_0\rho}\Big(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - k_y^2\Big)F\phi + \frac{1}{\mu_0\rho}\frac{\mathrm{d}^2 F}{\mathrm{d}x^2}F\phi, \tag{2.92}$$

where we have defined $F(x) = \boldsymbol{k}\cdot\boldsymbol{B}(x)$. After some manipulations, we obtain the following eigenmode equation;

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[(\omega^2 - \omega_{\mathrm{A}}^2)\frac{\mathrm{d}\phi}{\mathrm{d}x}\Big] - k_y^2(\omega^2 - \omega_{\mathrm{A}}^2)\phi = 0, \tag{2.93}$$

where $\omega_{\mathrm{A}}(x) = F(x)/\sqrt{\mu_0\rho}$.

The singular solution of the spectral equation (2.93) can be obtained as follows. Since Eq. (2.93) is a Sturmian equation, it should not have any singular solution except the Alfvén resonance ($\omega^2 - \omega_{\mathrm{A}}^2 = 0$). Suppose that $\omega^2 - \omega_{\mathrm{A}}^2(x)$ has the zero of order h ($\in \mathbb{N}$) at $x = x_{\mathrm{s}}$, i.e.

$$\omega^2 - \omega_{\mathrm{A}}^2(x) = c(x)(x - x_{\mathrm{s}})^h, \tag{2.94}$$

where $c(x)$ is an analytic function with finite value at $x = x_{\mathrm{s}}$. It is noted that, since the coefficient of the highest order derivative vanishes at $x = x_{\mathrm{s}}$, it constitutes a singular point of the spectral equation (2.93). For investigating the behavior of the solution in the vicinity of the singular point x_{s}, we will take the leading order of the Taylor expansion (2.94) and substitute it into Eq. (2.93), which yields

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \frac{h}{x - x_{\mathrm{s}}}\frac{\mathrm{d}\phi}{\mathrm{d}x} + k_y^2\phi = 0. \tag{2.95}$$

It is clearly seen that the point $x = x_{\mathrm{s}}$ is found to be a regular singular point of the spectral equation (2.93). The behavior of the solution around the singular point is investigated by means of the Frobenius expansion [16]. There is a logarithmic singularity in the solution since two solutions of the indicial equation have an integral difference for any $h \in \mathbb{N}$. Therefore, the solution is written in the vicinity of the regular singular point as

$$\phi(x) = a_1 g_1(x) + a_2[g_1(x)\log|x - x_{\mathrm{s}}| + g_2(x)], \tag{2.96}$$

where $g_1(x)$ and $g_2(x)$ are analytic functions with $g_1(x_s) \neq 0$. The energy norm (2.77) will be applied to the solution (2.96), which now reads as

$$\langle \phi \mid \phi \rangle = (\phi \mid -\Delta \mid \phi). \tag{2.97}$$

Thus, it is found that Eq. (2.96) actually gives a non square integrable solution corresponding to the continuous spectrum.

Furthermore, the fact that Eq. (2.96) is the only solution for the spectral equation (2.93) is shown as follows. Let $\underline{\omega_A^2}$ ($\overline{\omega_A^2}$) be the lower (upper) bound value of the Alfvén wave frequency

$$\underline{\omega_A^2} = \inf_{x \in \Omega} \omega_A^2 \quad (\overline{\omega_A^2} = \sup_{x \in \Omega} \omega_A^2), \tag{2.98}$$

if it exists in the plasma domain Ω. Here we divide the Alfvén singular factor as

$$\omega^2 - \omega_A^2 = (\omega^2 - \underline{\omega_A^2}) + (\underline{\omega_A^2} - \omega_A^2), \tag{2.99}$$

and multiply $\bar{\phi}$ denoting the complex conjugate of the stream function ϕ. Then, the integrated form of Eq. (2.93) gives

$$(\omega^2 - \underline{\omega_A^2}) \int_\Omega \left(\left| \frac{d\phi}{dx} \right|^2 + k_y^2 |\phi|^2 \right) dx = - \int_\Omega (\underline{\omega_A^2} - \omega_A^2) \left(\left| \frac{d\phi}{dx} \right|^2 + k_y^2 |\phi|^2 \right) dx. \tag{2.100}$$

Since we have taken the lower bound of ω_A^2 by $\underline{\omega_A^2}$, we see that $(\underline{\omega_A^2} - \omega_A^2) \leq 0$ at any position. Moreover, the large round bracket of the right hand side integrand denotes the local kinetic energy; i.e. $|\nabla\phi|^2 \geq 0$. Thus, it is shown that

$$- \int_\Omega (\underline{\omega_A^2} - \omega_A^2) \left(\left| \frac{d\phi}{dx} \right|^2 + k_y^2 |\phi|^2 \right) dx \geq 0. \tag{2.101}$$

The right hand side of Eq. (2.100) is shown to be positive and the integral of the left hand side is also positive, therefore

$$\omega^2 \geq \underline{\omega_A^2} \tag{2.102}$$

must hold. It is concluded that the Alfvén eigenmode equation (2.93) has no eigenvalue lower than the lower bound of the Alfvén continuous spectrum. If we trace the same discussion on the upper bound $\overline{\omega_A^2}$ of the Alfvén continuum, we can also prove that the Alfvén equation (2.93) does not have any eigenvalue upper than the upper bound of the continuum. Since it is quite natural to assume that $\omega_A(x)$ is a smooth function of x, we may conclude that the slab Alfvén equation has only Alfvén continuous spectrum. The spectra of Eq. (2.93) is shown as

$$\sigma_c = \{\omega^2 \mid \min_{x \in \Omega} \omega_A^2 \leq \omega^2 \leq \max_{x \in \Omega} \omega_A^2\}. \tag{2.103}$$

It is noted here that a meticulous care should be taken for the norm of the system. For example, in determining the square integrability of the solution (2.96), the energy norm (2.97) plays an essential role. If we take a simple norm $(\mid)$ here, then we find that the solution (2.96) does not give non square integrability since the square of the logarithmic function is integrable around the singular point $x = x_{\mathrm{s}}$. Furthermore, if we write the equation for the artificially introduced variable $\phi^\dagger = (\omega^2 - \omega_{\mathrm{A}}^2)\phi$ as

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[(\omega^2 - \omega_{\mathrm{A}}^2)\frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{\phi^\dagger}{\omega^2 - \omega_{\mathrm{A}}^2}\Big)\Big] - k_y^2\phi^\dagger = 0, \tag{2.104}$$

in order to eliminate the singularity from the equation, and take the simple norm $(\mid)$ again, we find that even the Alfvén equation can be rewritten in apparently non-Hermitian form. In this case, we have to take the norm as

$$\langle \phi^\dagger \mid \phi^\dagger \rangle = -\int \frac{\bar{\phi}^\dagger}{\omega^2 - \omega_{\mathrm{A}}^2}\,\Delta\Big(\frac{\phi^\dagger}{\omega^2 - \omega_{\mathrm{A}}^2}\Big)\,\mathrm{d}x, \tag{2.105}$$

which recovers the original Hermiticity of the system and the non square integrability of the solution.

It is also noted that the shear Alfvén continuum is not the only continuum in the MHD system. Their existence is first conjectured by Grad [74]. Firstly, he conjectured four branches of such continuum, however, it is clarified later by Appert *et al.* that the MHD system contains just two [36, 69]; one is the above shear Alfvén wave continuum, and the other is related to the sound wave.

2.6.2 Cylindrical geometry

Here the equilibrium magnetic field is assumed as

$$\boldsymbol{B} = (0, B_\theta(r), B_z), \tag{2.106}$$

with $B_z = \text{const}$ in the cylindrical coordinates. Then, we may take the wave number vector $\boldsymbol{k} = (0, m/r, k_z)$ due to the homogeneity of the equilibrium fields in the θ and z direction. In the same way as in the slab geometry, we replace ∂_t by $-\mathrm{i}\omega$. Then, Eq. (2.89) becomes

$$-\omega^2\Big[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r\frac{\mathrm{d}}{\mathrm{d}r}\Big) - \frac{m^2}{r^2}\Big]\phi = -F\Big[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r\frac{\mathrm{d}}{\mathrm{d}r}\Big) - \frac{m^2}{r^2}\Big]F\phi + \frac{m}{r}\frac{\mathrm{d}j}{\mathrm{d}r}F\phi, \tag{2.107}$$

where $F(r) = \boldsymbol{k}\cdot\boldsymbol{B}(r)$. Since the relation between the current density and $F(r)$ differs from the slab geometry due to the curvature effect, we will obtain the following spectral equation which is different from that in the slab geometry [Eq. (2.93)]

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big[r(\omega^2 - \omega_{\mathrm{A}}^2)\frac{\mathrm{d}\phi}{\mathrm{d}r}\Big] - \frac{m^2}{r^2}(\omega^2 - \omega_{\mathrm{A}}^2)\phi + \frac{2}{r}\frac{\mathrm{d}F}{\mathrm{d}r}F\phi = 0, \tag{2.108}$$

where, $\omega_{\mathrm{A}}(r) = F(r)/\sqrt{\mu_0 \rho}$.

Singular solution may be also obtained from the singularity of the equation;

$$\omega^2 = \omega_{\mathrm{A}}^2(r) \quad (\exists r \in \Omega). \tag{2.109}$$

It is noted that, however, this singularity is no longer regular due to the existence of the last term of Eq. (2.108). Therefore, we do not have general explicit representation of the singular solution. Moreover, the last term in Eq. (2.108) admits the point spectra which is the essential difference from the case in the slab geometry. Namely, even if ω^2 is less than the lower bound of the Alfvén wave frequency $\underline{\omega_{\mathrm{A}}^2} = \inf \omega_{\mathrm{A}}^2$,

$$\begin{aligned}(\omega^2 - \underline{\omega_{\mathrm{A}}^2}) \int_\Omega r \left(\left| \frac{\mathrm{d}\phi}{\mathrm{d}r} \right|^2 + \frac{m^2}{r^2} |\phi|^2 \right) \mathrm{d}r \\ = - \int_\Omega r(\underline{\omega_{\mathrm{A}}^2} - \omega_{\mathrm{A}}^2) \left(\left| \frac{\mathrm{d}\phi}{\mathrm{d}r} \right|^2 + \frac{m^2}{r^2} |\phi|^2 \right) \mathrm{d}r + \int_\Omega 2F \frac{\mathrm{d}F}{\mathrm{d}r} |\phi|^2 \, \mathrm{d}r,\end{aligned} \tag{2.110}$$

can be satisfied for an appropriate F satisfying

$$\int_\Omega 2F \frac{\mathrm{d}F}{\mathrm{d}r} |\phi|^2 \, \mathrm{d}r = - \int_\Omega r(\omega^2 - \omega_{\mathrm{A}}^2) \left(\left| \frac{\mathrm{d}\phi}{\mathrm{d}r} \right| + \frac{m^2}{r^2} |\phi|^2 \right) \mathrm{d}r \geq 0 \tag{2.111}$$

for a certain nontrivial eigenfunction ϕ.

Due to Sturm's oscillation theorem [16], if the solution for $\omega^2 = 0$ satisfying the boundary condition only on $r = 0$ have any node in the domain Ω, we will have unstable eigenvalue $\omega^2 < 0$ which also satisfies the boundary condition on the plasma edge. Furthermore, the number of these point spectra are infinite, which has a property to accumulate on the edge of the continuum $\omega^2 = \underline{\omega_{\mathrm{A}}^2}$. This property can be used in order to judge the stability of the resonant mode which has the edge of the continuum at $\underline{\omega_{\mathrm{A}}^2} = 0$. When the smallest eigenvalue is positive and the mode has no resonant surface inside the plasma, the eigenfunction shows a global stable oscillation, which is called the global Alfvén eigenmode [35].

2.7 Non-Hermiticity in shear flow systems

We will consider the shear flow introduced non-Hermiticity in this section. For simplicity, we will assume that the plasma is not magnetized and consider the electrostatic response. If we neglect the charge separation of the plasma, the plasma behaves in the same way as the neutral fluids. The equilibrium velocity field is assumed as

$$\boldsymbol{v}_0 = (0, v_{0y}(x), 0), \tag{2.112}$$

in the Cartesian coordinates. Then, the vorticity equation (2.66) for two dimensional incompressible motion of the plasma will be rewritten in the form of Rayleigh equation as

$$(\partial_t + v_{0y}\partial_y)\Delta\phi - v_{0y}''\partial_y\phi = 0. \tag{2.113}$$

Here, the prime denotes the derivative with respect to x and $\Delta = \partial_x^2 + \partial_y^2$. Assuming two dimensional perturbation, the wave number in the y direction becomes a good quantum number. Equation (2.113) will be written in the form of the Schrödinger type as

$$\mathrm{i}\partial_t\Delta\phi = k_y v_{0y}\Delta\phi - k_y v_{0y}''\phi. \tag{2.114}$$

We can play with Eq. (2.114) on the definition of the norm. If we regard Eq. (2.114) as a vorticity evolution equation, the generator is written as

$$\mathcal{L} = k_y v_{0y} - k_y v_{0y}''\Delta^{-1}. \tag{2.115}$$

Firstly, let us see how the simple norm for the vorticity field works. Suppose that the norm is defined by the enstrophy bilinear form;

$$\langle\!\langle \varPsi \mid \varPsi \rangle\!\rangle = (\Delta\phi \mid \Delta\phi) = \int |\Delta\phi|^2 \,\mathrm{d}V, \tag{2.116}$$

where we have defined $\varPsi = -\Delta\phi$. Then, the first operator in Eq. (2.115) trivially gives Hermiticity as

$$\begin{aligned}\langle\!\langle \varPsi \mid k_y v_{0y}\varPsi^* \rangle\!\rangle &= \int k_y v_{0y}\bar{\varPsi}\varPsi^* \,\mathrm{d}V \\ &= \langle\!\langle k_y v_{0y}\varPsi \mid \varPsi^* \rangle\!\rangle. \end{aligned}\tag{2.117}$$

The second operator gives non-Hermiticity with the enstrophy norm;

$$\begin{aligned}\langle\!\langle \varPsi \mid k_y v_{0y}''\Delta^{-1}\varPsi^* \rangle\!\rangle &= \langle\!\langle \Delta^{-1}k_y v_{0y}''\varPsi \mid \varPsi^* \rangle\!\rangle \\ &\neq \langle\!\langle k_y v_{0y}''\Delta^{-1}\varPsi \mid \varPsi^* \rangle\!\rangle. \end{aligned}\tag{2.118}$$

However, if we define the energy norm as

$$\begin{aligned}\langle \Delta\phi \mid \Delta\phi \rangle &= -\int (\Delta\bar{\phi})\,\Delta^{-1}(\Delta\phi)\,\mathrm{d}V \\ &= \int |\nabla\phi|^2 \,\mathrm{d}V, \end{aligned}\tag{2.119}$$

then, the second operator in Eq. (2.115) gives Hermiticity as

$$\begin{aligned}\langle \varPsi \mid k_y v_{0y}''\Delta^{-1}\varPsi^* \rangle &= -\int \bar{\varPsi}\,\Delta^{-1}(k_y v_{0y}''\Delta^{-1}\varPsi^*)\,\mathrm{d}V \\ &= -\int (k_y v_{0y}''\Delta^{-1}\bar{\varPsi})\,\Delta^{-1}\varPsi^* \,\mathrm{d}V \\ &= \langle k_y v_{0y}''\Delta^{-1}\varPsi \mid \varPsi^* \rangle, \end{aligned}\tag{2.120}$$

whereas the first operator gives non-Hermiticity;

$$\begin{aligned}
\langle \Psi \,|\, k_y v_{0y} \Psi^* \rangle &= -\int \bar{\Psi}\, \Delta^{-1}(k_y v_{0y} \Psi^*)\, \mathrm{d}V \\
&= -\int (k_y v_{0y} \Delta^{-1} \bar{\Psi})\, \Delta\Delta^{-1} \Psi^* \, \mathrm{d}V \\
&= -\int (\Delta k_y v_{0y} \Delta^{-1} \bar{\Psi})\, \Delta^{-1} \Psi^* \, \mathrm{d}V \\
&= \langle \Delta k_y v_{0y} \Delta^{-1} \Psi \,|\, \Psi^* \rangle. \qquad (2.121)
\end{aligned}$$

Let us then regard Eq. (2.114) as an evolution equation for the stream function ϕ;

$$\mathrm{i}\partial_t \phi = k_y \Delta^{-1} v_{0y} \Delta\phi - k_y \Delta^{-1} v''_{0y} \phi. \qquad (2.122)$$

Then, we can readily show that the first operator is non-Hermitian with the energy norm

$$\begin{aligned}
\langle \phi \,|\, \phi \rangle &= -\int \bar{\phi}\, \Delta\phi \, \mathrm{d}V \\
&= \int |\nabla\phi|^2 \, \mathrm{d}V, \qquad (2.123)
\end{aligned}$$

however, it is Hermitian with the enstrophy norm

$$\begin{aligned}
\langle\!\langle \phi \,|\, \phi \rangle\!\rangle &= \int \bar{\phi}\, \Delta^2\phi \, \mathrm{d}V \\
&= \int |\Delta\phi|^2 \, \mathrm{d}V. \qquad (2.124)
\end{aligned}$$

On the other hand, the second operator is Hermitian with the energy norm and non-Hermitian with the enstrophy norm. It is, therefore, concluded that the Hermiticity of the operator does not change by the form of the evolution equation if we take the common norm.

It has been shown that neither energy norm nor enstrophy norm gives Hermiticity of the combined operator for the vorticity evolution equation (2.114). However, when $v''_{0y} \neq 0$ in the domain, we can make them Hermitian by taking the norm

$$\langle\!\langle\!\langle \phi \,|\, \phi \rangle\!\rangle\!\rangle = \int \frac{1}{|v''_{0y}|} \bar{\phi}\, \phi \, \mathrm{d}V. \qquad (2.125)$$

It is straightforwardly shown that the combined operator is Hermitian with the norm (2.125) as

$$\begin{aligned}
\langle\!\langle\!\langle \Psi \,|\, k_y(v_{0y} - v''_{0y}\Delta^{-1})\Psi^* \rangle\!\rangle\!\rangle &= \int \frac{k_y v_{0y}}{v''_{0y}} \bar{\Psi}\Psi^* - \bar{\Psi}\, k_y \Delta^{-1} \Psi^* \, \mathrm{d}V \\
&= \int \frac{k_y}{v_{0y''}} (v_{0y} - v''_{0y}\Delta^{-1}) \bar{\Psi}\, \Psi^* \, \mathrm{d}V \\
&= \langle\!\langle\!\langle k_y(v_{0y} - v''_{0y}\Delta^{-1})\Psi \,|\, \Psi^* \rangle\!\rangle\!\rangle, \qquad (2.126)
\end{aligned}$$

where we have assumed that $v''_{0y} > 0$ for simplicity. In this case, the spectra of the generator $k_y(v_{0y} - v''_{0y}\Delta^{-1})$ are real, which yields stability of the system. This fact shows that the shear flow is stable when the system does not contain any inflection point (Rayleigh's inflection point theorem [111, 7]).