Chapter 6

Interchange instabilities of slab plasmas with sheared flows

6.1 Introduction

It is widely accepted that a shear flow yields stabilizing effects on various fluctuations through convective deformations of disturbances [97, 99, 136]. However, rigorous treatment of the shear flow effects encounters a serious difficulty arising from the non-Hermiticity of the problem. We may not consider well-defined 'modes' and corresponding 'time constants.' The standard normal mode approach breaks down, and the theory may fail to give correct predictions of evolution even if perturbed fields remain in the linear regime. The discrepancies between the theory and the experiment on the stability limit of neutral fluids are reviewed in Ref. [140, 115]. The aim of this work is to establish a solid foundation for the analysis of shear flow systems. We apply Kelvin's method of shearing modes [139]. This scheme, previously called as 'nonmodal' approach, actually consists in the combination of two methods which have been widely used in solving wave equations; the modal and the characteristics methods.

Many works have been done on instability problems of plasmas with shear flows by means of the simple 'modal' approach. It is implicitly assumed in the application of the modal scheme that the motion can be decomposed into a set of independent 'normal modes' with certain time constants [77, 43]. As is well-known, this method is effective in solving problems involving Hermitian operators. However, when applying it to non-Hermitian systems, we may overlook the secular and transient behaviors. On the other hand, the characteristics method has been used in the context of rapid distortion theory for the fluid turbulence [114] and in the eikonal representation of the ballooning mode stability [79]. If we can treat the non-Hermitian part of the whole operator as a singular perturbation to the Hermitian operator [66, 68, 151], we may be able to construct a theory in the framework of the perturbation theory for the operator [18]. But unfortunately the convergence of the perturbative series seems to be very ambiguous in case of the shear flows due to the secularity of their time evolutions [33, 84, 152]. Thus, a thorough mathematical treatment of the non-Hermitian operators of shear flow systems has not been accomplished so far. In this chapter, we have analyzed the shear flow effect on interchange instabilities and its non-Hermitian mathematical background with the time asymptotic behavior by means of Kelvin's method.

Recently, Kelvin's method has been applied to a variety of linear shear flow problems [57, 52, 53, 112, 100, 137, 143]. For neutral and magnetized fluids, many new fascinating phenomena were discovered; exchanges of energy between background flows and perturbed fields [53], shear flow induced coupling between sound waves and internal waves and the excitation of beat wave [112], the asymptotic persistence due to the periodic energy transfer for two dimensional shear flows [100], and the emission of magnetosonic waves by the stationary vortex perturbations [137]. These results show that the modes, which are independent for static fluids, are no longer independent and the coupling of these modes induces the energy transfer in the presence of the shear flow. The basic properties of kink-type instabilities in the presence of a background shear flow is also analyzed [143]. It is shown that the shear flow mixing always overcomes the kink driving at sufficiently large time. However, the mathematical significance of this method has not been clarified yet.

In this chapter, we will first revisit Kelvin's method from the viewpoint of the characteristics method in Sec. 6.2. We will review the spectral theory focusing on the general mathematical concept of eigenmode for a better understanding of Kelvin's method. In Sec. 6.3, we will give the equations governing the interchange instabilities. In Sec. 6.4, we will derive the ordinary differential equation (ODE) for the time evolution of the amplitude of the interchange instabilities by applying the analysis of shearing modes. In Sec. 6.5, by drawing an analogy with Newton's equation it will be shown that the solution to the above ODE for the flux function exhibits an asymptotic damped behavior for any strength of instability drive. We will also consider the electrostatic perturbations in Sec. 6.6. Here the solution of the derived ODE for the stream function shows the asymptotic growth or decay of algebraic type depending on the magnitude of instability drive. The difficulty encountered by including the magnetic shear in the present formulation is addressed in Sec. 6.7. We will summarize the obtained results in Sec. 6.8.

6.2 Non-Hermiticity of shear flow systems

Before formulating the interchange instability equations, let us give a rough sketch of the problem and explain the mathematical tool to analyze the non-Hermitian dynamics. As is well known, the force operator governing the linear dynamics of static MHD plasmas is Hermitian [10], and therefore the perturbed fields can be decomposed into a set of orthogonal eigenmodes which show purely exponential (unstable) or purely oscillating (stable) evolutions. A non-triviality stems from the Alfvénic and acoustic continuous spectra; the phase mixing damping occurs. This behavior, however, is totally within the framework of the well-known theory of Hermitian operators due to von Neumann [28].

In the case where ambient the shear flow exists, however, the operator becomes non-Hermitian and the resolution in terms of orthogonal eigenmodes fails. From a dynamical point of view, the system experiences a complex type of non-exponential evolutions. In the following sections, we will show examples of such kind of 'non-Hermitian' dynamics where transient phenomena and secular evolutions play a dominant role. Similar evolutions are found in the case of non-Hermitian operators in finite dimensional vector spaces [61]. It is emphasized that the application of the traditional modal paradigm to non-Hermitian systems, which assumes exponential evolution of the perturbed fields, hinders these rich varieties of transient and algebraic phenomena. In this section, we will discuss Kelvin's method and show its suitability to the analysis of shear flow induced non-Hermitian systems. We will revisit it from the viewpoint of the characteristics method showing that it represents a generalization of the modal approach.

The linearized dynamics of fluid systems in the presence of sheared flow is governed by a general equation of the following type;

$$\partial_t u + \boldsymbol{v} \cdot \nabla u = \mathcal{A} u, \tag{6.1}$$

where $\mathcal{A}$ denotes a Hermitian differential operator (time-independent) defined in a Hilbert space V, $\boldsymbol{v}$ is the stationary mean flow, and u ($\in V$) denotes a perturbed field.

It is the convective derivative, $\boldsymbol{v} \cdot \nabla$, that introduces the non-Hermiticity into problem (6.1) and prevents the possibility of representing the dynamics of the systems in terms of orthogonal and complete set of eigenfunctions. This is a well known difficulty in the stability analysis of neutral fluids, such as Couette or Poiseille flows, where the predictions obtained by means of the modal methods do not match the experiments [140, 115].

In the case of a spatially inhomogeneous stationary flow $\boldsymbol{v}$, Eq. (6.1) becomes non-Hermitian and a straightforward spectral resolution is not effective. However, Kelvin's method permits to resolve, for some classes of mean flows, the evolution of the system (6.1) into new types of modes by means of which both transient and secular asymptotic behaviors are effectively described. Let us now explain mathematical foundations of this scheme.

As mentioned in Sec. 6.1, Kelvin's method consists in the combined application of two methods which have been extensively used in the analysis of wave equations. Precisely the 'Lagrangian' part of Eq. (6.1), $\partial_t + \boldsymbol{v} \cdot \nabla$, is solved by means of the characteristics method and the 'Hermitian' part $\mathcal{A}$ by means of the standard spectral resolution.

The characteristics method is applied to solve the characteristic ODE associated to the Lagrangian derivative, which is moving along the characteristic curve of the ambient motion, given by

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{v}, \quad \boldsymbol{x}(0) = \boldsymbol{\xi}. \tag{6.2}$$

By inverting the modes, which are expressed in the Lagrangian coordinates as $\varphi(\boldsymbol{k}, \boldsymbol{\xi})$, they will be represented in the Eulerian coordinates as

$$\tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x}) = \varphi(\boldsymbol{k}, \boldsymbol{\xi}(t; \boldsymbol{x})), \tag{6.3}$$

where $\boldsymbol{\xi}(t; \boldsymbol{x})$ denotes the inverse of $\boldsymbol{x}(t; \boldsymbol{\xi})$. The existence of the inverse mapping $\boldsymbol{x}(t) \mapsto \boldsymbol{\xi}$ is guaranteed in the case of incompressible mean flows. Due to Eq. (6.3), $\tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x})$ satisfies the characteristic equation

$$\partial_t \tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x}) + \boldsymbol{v} \cdot \nabla \tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x}) = 0. \tag{6.4}$$

The essential condition for the applicability of Kelvin's method consists in the constraint for the functions $\tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x})$ to form the complete set of eigenfunctions of the operator $\mathcal{A}$. If such a set of eigenfunctions exists, we can decompose the perturbed field u by means of

$$u = \int \tilde{u}_k(t)\, \tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x})\, \mathrm{d}\boldsymbol{k}. \tag{6.5}$$

We notice that due to Eq. (6.3) the eigenvalues of $\mathcal{A}$ become time dependent. The new eigenvalue problem for $\mathcal{A}$ reads

$$\mathcal{A}\tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x}) = \lambda_k(t)\, \tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x}). \tag{6.6}$$

Plugging Eq. (6.5) into Eq. (6.1) and exploiting Eqs. (6.4) and (6.6), we have

$$\int [\partial_t \tilde{u}_k(t)]\, \tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x})\, \mathrm{d}\boldsymbol{k} = \int \tilde{u}_k(t) \lambda_k(t)\, \tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x})\, \mathrm{d}\boldsymbol{k}. \tag{6.7}$$

Due to the orthogonality of the modes $\tilde{\varphi}(t;\boldsymbol{k},\boldsymbol{x})$, the evolution of $\tilde{u}_k$ is governed by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{u}_k(t) = \lambda_k(t)\,\tilde{u}_k(t). \tag{6.8}$$

If $\tilde{\varphi}(t;\boldsymbol{k},\boldsymbol{x})$ do not satisfy both conditions given by the characteristic equation (6.4) and the eigenvalue equation (6.6), Eq. (6.7) includes additional terms representing the complicated mode coupling. Thus, the applicability of Kelvin's method is compromised in this case.

Due to the time dependence in the eigenvalues $\lambda_k(t)$, the evolution of $\tilde{u}_k(t)$ will not exhibit a simple exponential dependence as in the Hermitian case. More complicated behaviors appear, which are characteristic of non-Hermitian systems. By analyzing the ODE (6.8), the motion of each mode can be classified, and the time asymptotic behavior can be also shown. The following sections will be devoted to the derivation of ODE (6.8) and the discussion of the behavior of its solution for the case of interchange instabilities in plasmas with shear flow.

6.3 Formulation of interchange instabilities

Interchange instabilities have been analyzed for static (ambient flow $\boldsymbol{v}_0 = 0$) magnetized plasmas by many authors [40, 96, 127, 103, 133]. In the case of static plasmas, the ideal MHD equations can be reduced into a simple partial differential equation of the form [10]

$$\partial_t^2 \boldsymbol{\xi} = \mathcal{F}\boldsymbol{\xi}, \tag{6.9}$$

where $\boldsymbol{\xi}$ is the displacement vector and $\mathcal{F}$ is the force operator which is Hermitian (selfadjoint) when the plasma is surrounded by an ideal conducting wall. In order to analyze the stability of the system, we can apply the spectral method and represent the dynamics in terms of a superposition of harmonic oscillations of normal modes. Another method of analyzing the stability of the static magnetized plasmas is to apply the energy principle [40, 96] which is a variational approach based on the Hermiticity of the force operator $\mathcal{F}$. These methods show that the unstable interchange modes have extremely spatially localized structures near the marginal stability [127, 103] except when $p' \simeq 0$ on the rational surface [133] (see Chap. 3).

It is remarkably difficult to estimate the exact linear stability of the system in the presence of a stationary shear flow, since, as seen in the previous sections, the dynamics become non-Hermitian and both the spectral and the variational methods lose their mathematical foundations. Dispersion relations have been studied in many publications [66, 97, 77, 43], however, as discussed in Sec. 6.1, the evolution of non-Hermitian system cannot be reconstructed from the formal dispersion relation, because we do not have a spectral theory. Since the proper asymptotic behavior of interchange instabilities are not yet clearly shown in the presence of shear flow, we will first analyze simplified systems focusing on the non-Hermiticity of the system. In this section, we will derive the governing equations for magnetized plasmas with stationary shear flows. Specifically, we will investigate the effect of shear flow on interchange instabilities in plasmas with an ambient homogeneous magnetic field.

In the presence of gravitational force, the ideal incompressible MHD equations are written as

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{j} \times \boldsymbol{B} - \nabla p + \rho \boldsymbol{g}, \tag{6.10}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \nabla \cdot \boldsymbol{v} = 0, \tag{6.11}$$

$$\partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = \boldsymbol{0}, \tag{6.12}$$

$$\nabla \cdot \boldsymbol{v} = 0, \tag{6.13}$$

where ρ, $\boldsymbol{B}$, and $\boldsymbol{g}$ are the density, magnetic field, and gravitational constant vector, and $\mathrm{d}/\mathrm{d}t = \partial_t + \boldsymbol{v} \cdot \nabla$ denotes the Lagrangian derivative. Here we assume the incompressibility of the velocity field $\boldsymbol{v}$, instead of using the equation of state.

The ambient fields (denoted by the subscript 0) must satisfy

$$\rho_0 \boldsymbol{v}_0 \cdot \nabla \boldsymbol{v}_0 = \boldsymbol{j}_0 \times \boldsymbol{B}_0 - \nabla p_0 + \rho_0 \boldsymbol{g}. \tag{6.14}$$

If we consider a parallel stationary shear flow of the form $\boldsymbol{v}_0 = (0, v_y(x), 0)$, straight homogeneous magnetic field $\boldsymbol{B}_0 = (0, B_y, B_z)$, and gravitational force acting in the positive x direction, the convective derivative gives no contribution to the stationary state and Eq. (6.14) is reduced to

$$\nabla p_0 = \rho_0 \boldsymbol{g}. \tag{6.15}$$

The above equation denotes that the pressure gradient is balanced by the gravitational force in the x direction. This is the same condition which holds for static neutral fluids.

The perturbed magnetic and velocity fields are assumed to be two dimensional in the xy plane, and thus we can introduce the poloidal flux function and stream function;

$$\begin{aligned} \boldsymbol{B}_{1\perp} &= \nabla \psi \times \boldsymbol{e}_z, \\ \boldsymbol{v}_{1\perp} &= \nabla \phi \times \boldsymbol{e}_z, \end{aligned} \tag{6.16}$$

where the subscript 1 denotes the perturbed quantities, $\perp$ expresses the direction perpendicular to the dominant magnetic field directed along the z axis, and $\boldsymbol{e}_z$

denotes the unit vector in the z direction. Using these representations, we can eliminate the pressure from governing equations.

Taking the curl of the equation of motion and projecting it along $\boldsymbol{e}_z$, we obtain

$$\mu_0\rho_0[(\partial_t + v_y\partial_y)\Delta\phi - v_y''\partial_y\phi] = \boldsymbol{B}_0 \cdot \nabla(\Delta\psi) + \mu_0\partial_y\rho_1 g, \tag{6.17}$$

where $\Delta = \partial_x^2 + \partial_y^2$. In deriving Eq. (6.17) we have used the Boussinesq approximation which neglects the spatial variation of the stationary state density in the inertial term of equation of motion, but does not in continuity equation, since it is the driving term for the interchange instability. Physically it is valid provided that the variability in the density is due to variations in the temperature of only moderate amounts [7]. The component of the flow perpendicular to the ambient magnetic field can be considered consistently coming from the $\boldsymbol{E} \times \boldsymbol{B}$ drift, taking into account the ideal Ohm's law. It is noted that, if we neglect the effect of the magnetic field, we recover Rayleigh equation for Kelvin-Helmholtz instability of the incompressible neutral fluid [9].

The density fluctuation can be expressed as

$$(\partial_t + v_y\partial_y)\rho_1 = -\rho_0'\partial_y\phi. \tag{6.18}$$

where the prime denotes the derivative with respect to x. Now ρ_0' is considered as a constant which introduces a destabilizing force. The induction equation is the same as in the ordinary reduced MHD equations [123, 124]. Under the above assumptions on the stationary fields, it reads as

$$(\partial_t + v_y\partial_y)\psi = \boldsymbol{B}_0 \cdot \nabla\phi. \tag{6.19}$$

Equations (6.17)-(6.19) constitute a closed system of equations. It is seen that the static system ($v_y = 0$) governed by these equations is Hermitian. It is the convective derivative ($v_y \neq 0$) that brings the non-Hermiticity into our system. Actually, the system of equations (6.17)-(6.19) can be obtained directly by replacing $g = 2p/\rho R_0$ in the high β reduced MHD equations describing tokamak plasmas [124], where R_0 denotes the major radius of the toroidal device. We will investigate the effect of the shear flow on the interchange instabilities in following sections.

6.4 Derivation of ordinary differential equation for Kelvin's mode

In this section, we derive the ordinary differential equation (ODE) for the amplitude of Kelvin's mode, given in Eq. (6.8), in the case of interchange instabilities of plasmas. Let us first consider the electromagnetic case where $\boldsymbol{B}_0 \cdot \nabla \neq 0$. From Eqs. (6.18)-(6.19), we have

$$\phi = -\partial_y^{-1} \rho_0'^{-1} (\partial_t + v_y \partial_y) \rho_1 = (\boldsymbol{B}_0 \cdot \nabla)^{-1} (\partial_t + v_y \partial_y) \psi. \tag{6.20}$$

Since we have assumed the mean velocity $v_y = v_y(x)$ and the homogeneous ambient field $\boldsymbol{B}_0 = (0, B_y, B_z)$, the operator $\partial_t + v_y \partial_y$ commutes with both ∂_y^{-1} and $(\boldsymbol{B}_0 \cdot \nabla)^{-1}$. Thus acting on both sides of Eq. (6.20) with the operator $(\partial_t + v_y \partial_y)^{-1}$ gives

$$\rho_1 = -\rho_0' \partial_y (\boldsymbol{B}_0 \cdot \nabla)^{-1} \psi. \tag{6.21}$$

From Eq. (6.19),

$$\Delta \phi = \Delta (\boldsymbol{B}_0 \cdot \nabla)^{-1} (\partial_t + v_y \partial_y) \psi. \tag{6.22}$$

Substituting Eqs. (6.20) and (6.22) into Eq. (6.17), and acting with $\boldsymbol{B}_0 \cdot \nabla$ on both sides, we obtain

$$(\partial_t + v_y \partial_y) \Delta (\partial_t + v_y \partial_y) \psi = \frac{(\boldsymbol{B}_0 \cdot \nabla)^2}{\mu_0 \rho_0} \Delta \psi - \frac{\rho_0' g}{\rho_0} \partial_y^2 \psi. \tag{6.23}$$

It is noted that the linearity of the ambient velocity profile allows us to eliminate v_y''.

Since the operator on the right hand side is Hermitian, we can decompose the flux function ψ by means of the shearing eigenmodes

$$\psi(\boldsymbol{x}, t) = \int \tilde{\psi}_k(t) \, \tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{k}, \tag{6.24}$$

where each eigenmode can be expressed by the sinusoidal function in our simplified case

$$\begin{aligned} \tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x}) &= \exp[\mathrm{i}k_x x + \mathrm{i}k_y (y - v_y t) + \mathrm{i}k_z z] \\ &= \exp[\mathrm{i}\tilde{k}_x(t) x + \mathrm{i}k_y y + \mathrm{i}k_z z]. \end{aligned} \tag{6.25}$$

Here the mean flow is assumed to be $v_y(x) = \sigma x$ and $\tilde{k}_x(t) = k_x - k_y \sigma t$. It is explicitly shown that the wave number in the flow shear direction is linearly increasing with time by the distortion of perturbation due to the sheared mean flow. However, the completeness of the modes $\tilde{\varphi}$ in the Hilbert space will not be lost due to the time dependent wave number $\tilde{k}_x$, therefore, the expansion (6.24) still gives a general solution of the system. Since continuous variation of $\tilde{k}_x(t)$ prevents from imposing the boundary condition in the bounded domain, we will concentrate on the analysis of localized perturbations by considering the infinite domain. Note that $\tilde{\varphi}$ are the eigenfunctions of the right hand side of Eq. (6.23), and also satisfy the characteristic equation (6.4). It should be noted that the presence of the Laplacian operator in

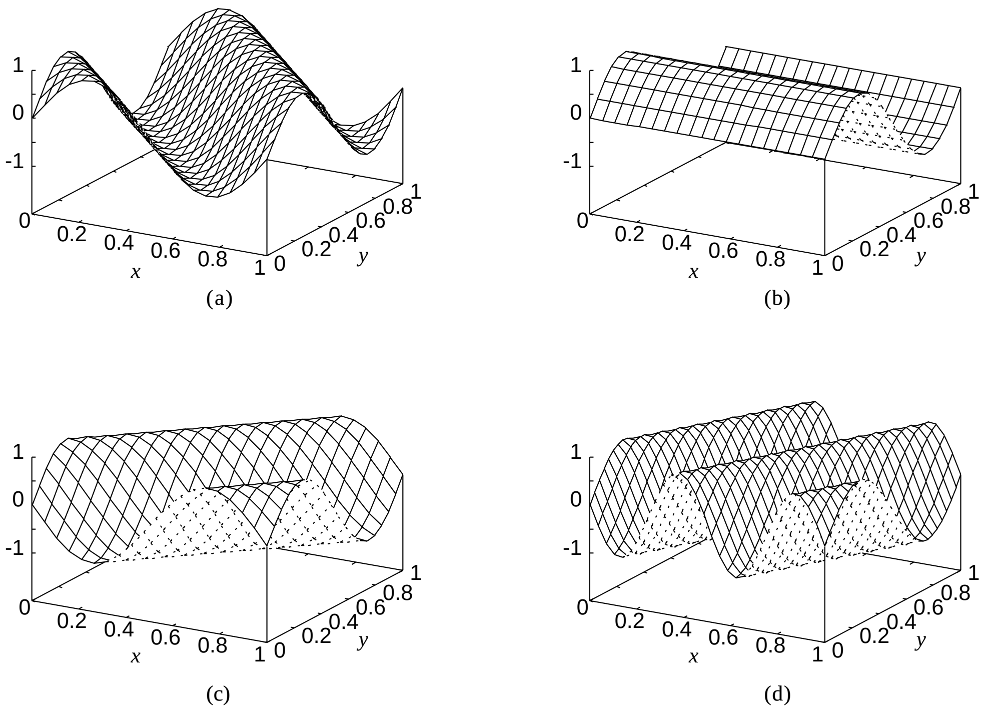

Figure 6.1: Kelvin's mode $\tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x})$.

the left hand side of Eq. (6.23) does not hinder the application of Kelvin's method since the modes $\tilde{\varphi}$ are as well eigenfunctions of the Laplacian Δ.

Thus, the time evolution equation for the amplitude $\tilde{\psi}_k$ can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\tilde{k}_x(t)^2 + k_y^2\right)\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t}\right] = -\frac{F^2}{\mu_0\rho_0}\left(\tilde{k}_x(t)^2 + k_y^2\right)\tilde{\psi} - k_y^2\frac{\rho_0' g}{\rho_0}\tilde{\psi}, \tag{6.26}$$

where $F = \boldsymbol{k}\cdot\boldsymbol{B}_0 = k_y B_{0y} + k_z B_{0z}$, and we have dropped the subscript k for simplicity. We notice that in the absence of shear flow ($\sigma = 0$) the usual interchange instability equation for static equilibrium can be obtained.

Our procedure can be readily shown to coincide with the traditional formulation of Kelvin's method consisting in the coordinate transform $(t, x, y, z) \mapsto (T, \xi, \eta, \zeta)$ defined by

$$T = t, \quad \xi = x, \quad \eta = y - \sigma t x, \quad \zeta = z, \tag{6.27}$$

and the Fourier transform with respect to the new coordinates

$$\tilde{u}(k_\xi, k_\eta, k_\zeta; T) = \iiint_{-\infty}^{+\infty} u(\xi, \eta, \zeta; T)\, e^{\mathrm{i}(k_\xi\xi + k_\eta\eta + k_\zeta\zeta)}\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\zeta. \tag{6.28}$$

Normalizing the time t by the poloidal Alfvén time $\tau_A = a\sqrt{\mu_0\rho_0}/F$, we can rewrite Eq. (6.26) in dimensionless form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\tilde{k}_x(t)^2 + k_y^2\right)\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t}\right] = -\left(\tilde{k}_x(t)^2 + k_y^2\right)\tilde{\psi} + k_y^2\frac{\tau_A^2}{\tau_G^2}\tilde{\psi}, \tag{6.29}$$

where the wave vectors are normalized by the characteristic length scale a and $\tau_G^2 = -\rho_0/\rho_0' g$. Further we can rewrite Eq. (6.29) in the form

$$\frac{\mathrm{d}^2\tilde{\psi}}{\mathrm{d}t^2} + \mu(t)\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t} + [1 - S(t)]\tilde{\psi} = 0, \tag{6.30}$$

where

$$\mu(t) = -\frac{2\sigma k_y \tilde{k}_x(t)}{\tilde{k}_x(t)^2 + k_y^2},$$
$$S(t) = \frac{k_y^2 G}{\tilde{k}_x(t)^2 + k_y^2},$$

and $G = \tau_A^2/\tau_G^2$. Drawing an analogy with Newton's equation, $\mu(t)$ represents the frictional term and $S(t)$ the interchange drive term. Equation (6.30) is the correspondent of Eq. (6.8). As we have mentioned in Sec. 6.2, the time evolution for the amplitude of each eigenmode is no longer described by a simple exponential function. The behavior of $\tilde{\psi}$ will be discussed in the following sections.

6.5 Asymptotic and transient behavior of Kelvin's mode

In the absence of a density gradient or shear flow, $\mu(t) = S(t) = 0$ in Eq. (6.30) and we have a pure oscillation representing the Alfvén wave. When we include the density gradient, then $S(t)$ becomes nonzero. Then, we obtain an exponentially growing interchange instability for negative ρ_0' which exceeds the threshold value. Since a homogeneous magnetic field is assumed here, we have no stabilizing effect of the magnetic shear. The operator is Hermitian in these two cases, therefore we have the simple exponential evolution with time constants for each mode.

When we include the shear flow, we have $\mu(t) \neq 0$ and we may consider an analogy for the dynamics of a damped oscillator with time dependent frictional coefficient $\mu(t)$. In the following subsections, we will describe both the asymptotic and transient behavior of the amplitude $\tilde{\psi}$.

6.5.1 Transient behavior

In this subsection, we will analyze the transient behavior of each perturbed mode. Since an analytic expression is not available, we discuss the time evolution by qualitatively analyzing the ODE (6.30). In the absence of the instability drive, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\Big(\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t}\Big)^2 + \tilde{\psi}^2\Big] = -\mu(t)\Big(\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t}\Big)^2, \tag{6.31}$$

Case	$\sigma k_x k_y$	$\mu(t=0)$	$\mu(t \to \infty)$
(a)	$-$	$+$	$+$
(b)	$+$	$-$	$+$

Table 6.1: The relation between the sign of quantity $\sigma k_x k_y$ and that of effective frictional coefficient $\mu(t)$.

where

$$\mu(t) = -\frac{2\sigma k_y \tilde{k}_x(t)}{\tilde{k}_x(t)^2 + k_y^2},$$
$$\tilde{k}_x(t) = k_x - \sigma k_y t.$$

Therefore, the frictional coefficient $\mu(t)$ acts as a damping term for $\mu > 0$. It may be considered that this damping is caused by mixing in the same way as Landau damping which is caused by shear flow in the phase space. On the other hand, if $\mu(t) < 0$, the oscillator will be amplified due to shear flow.

Since the sign of the denominator in $\mu(t)$ is always positive, the behavior of the solution will be determined by the sign the numerator

$$\mu(t) \propto -2\sigma k_y \tilde{k}_x(t) = -2\sigma k_x k_y + 2(k_y\sigma)^2 t. \tag{6.32}$$

It can be easily understood from Eq. (6.32) that $\mu(t)$ will certainly be positive for large t regardless of the sign of the wave number or the flow shear. Thus, we may conclude that the shear flow acts to damp the oscillation in a time asymptotic sense. For the negative product $\sigma k_x k_y$ [Table 6.1(a)], the frictional coefficient $\mu(t)$ is always positive, therefore, the mode will be damped from the beginning. On the other hand, for the positive product $\sigma k_x k_y$ [Table 6.1(b)], $\mu(t)$ is negative at first, and goes to positive through zero at the instant $t_* = k_x/\sigma k_y$. Therefore the mode experiences an initial amplification lasting until the time t_*, which is even faster than the case with the interchange drive only.

It is interesting to see the relations between the frictional coefficient and the wave vector. We will take here as $\sigma > 0$ and $k_y > 0$ for simplicity, and the same conclusion may be drawn if we change the corresponding signs appropriately in other cases. As is shown in Fig. 6.1, the eigenfunction is being distorted due to the stretching effect of the shear flow, and the direction of the corresponding wave vector is also shifted. The $|\tilde{k}_x(t)|$ of the mode with negative initial k_x [see Figs. 6.1(c) and 6.2(a)] will be increased monotonically, and its structure becomes finer and finer. Then the mixing is promoted and its amplitude is damped. On the other hand, the $|\tilde{k}_x(t)|$ of the mode with positive initial k_x [see Figs. 6.1(a) and 6.2(b)] will be decreased in $t < t_*$, its

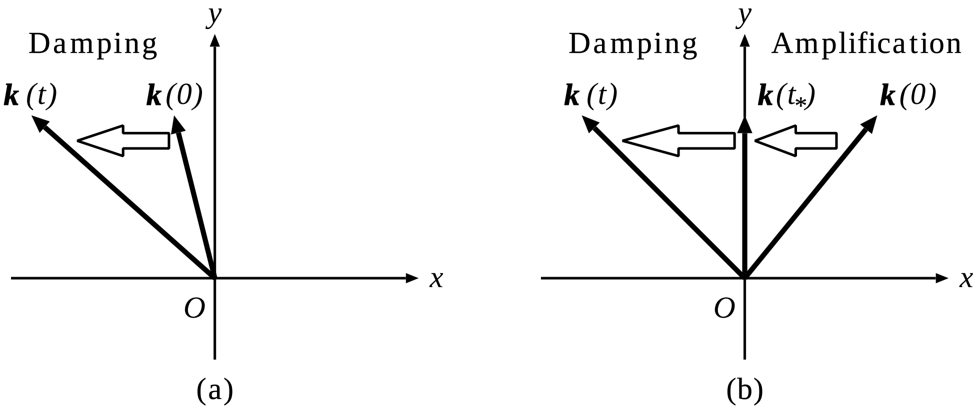

Figure 6.2: Distortion of the wave vector due to shear flow and its effect on the amplitude of magnetic flux.

structure once becomes the most coarse at time $t = t_*$, and becomes finer and finer. Thus the amplitude of the mode is amplified in the period $t < t_*$, and damped after that due to the mixing effect. One example of the numerical solutions of Eq. (6.30) is shown in Fig. 6.3. The result corresponds to the Case (b) of the Table 6.1. It is also seen in this figure that the initial amplification of the perturbation lasts until the turning point $t_* = 50$, then it is followed by the asymptotic decaying phase.

We have observed in the numerical solutions that there is a case where the amplitude is amplified to the value of 10^{30} times larger than the initial one. From a physical point of view, such huge amplifications may break down the linearity of the perturbations and may lead to a nonlinear stage. This case is beyond the scope of the linear theory and no sure conclusion can be drawn from Kelvin's method. Such huge amplifications are experienced by modes with large t_* and G.

6.5.2 Asymptotic behavior

In order to study the time asymptotic behavior, we assume $t \gg k_x/\sigma k_y,\ 1/\sigma$. In this time asymptotic limit we obtain the following ODE

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\tilde{\psi} + \frac{2}{t}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\psi} + \Big(1 - \frac{G/\sigma^2}{t^2}\Big)\tilde{\psi} = 0, \tag{6.33}$$

where $G = \tau_A^2/\tau_G^2$ denotes the magnitude of the instability drive term. In the absence of the instability drive G, the time asymptotic behavior of the solution of Eq. (6.33) is expressed as

$$\tilde{\psi} \sim \frac{1}{t}\sin t, \tag{6.34}$$

which coincides with the result of Koppel [92] who studied a time dependent non-perturbative state. Since Eq. (6.33) corresponds to the spherical Bessel equation,

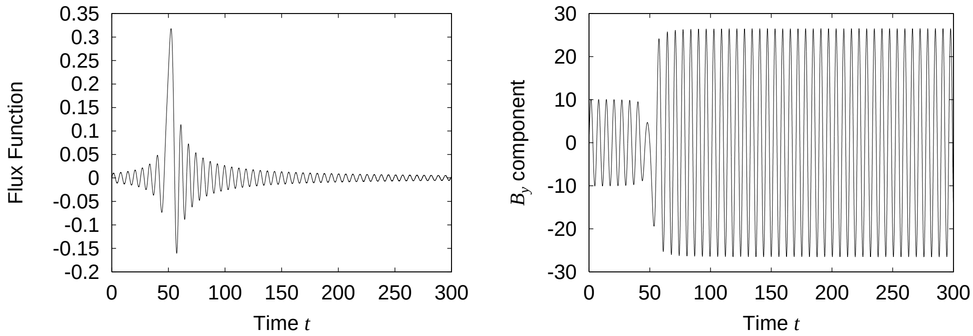

Figure 6.3: Numerical integrations of Eq. (6.30) for the parameters $k_x = 10$, $k_y = 1$, $k_z = 0$, $\sigma = 0.2$, and $G = 1$. Initial perturbations are $\tilde{\psi} = 0$ and $\mathrm{d}\tilde{\psi}/\mathrm{d}t = 1.0$ at $t = 0$.

its general solution for $G \neq 0$ is expressed as

$$\tilde{\psi} = \frac{1}{\sqrt{t}}(C_1 J_\nu(t) + C_2 Y_\nu(t)), \tag{6.35}$$

where J_ν and Y_ν denote the Bessel functions, and $\nu = (G/\sigma^2 + 1/4)^{1/2}$. Therefore the time asymptotic behavior of the mode is expressed generally as

$$\tilde{\psi} \sim \frac{1}{t}\sin\Big(t - \frac{\pi\nu}{2} + \delta\Big), \tag{6.36}$$

where δ denotes a constant phase depending on the initial condition. Therefore, the mode oscillates with amplitude $\tilde{\psi}$ and decays with the inverse power of time. While the x component of the perturbed magnetic field $\tilde{b}_x$ is proportional to ψ, the y component $\tilde{b}_y$ is proportional to $\tilde{k}_x(t)\tilde{\psi}$. Thus $\tilde{b}_y$ tends to the pure oscillatory behavior

$$\tilde{b}_y \sim \sin\Big(t - \frac{\pi\nu}{2} + \delta\Big), \tag{6.37}$$

as $\tilde{k}_x(t)$ increases with proportional to time (see Fig. 6.3). It should be noted that there is no threshold value for the stabilization of the interchange instability, since we obtain the same spherical Bessel equation (6.33) for all modes. All modes asymptotically evolve by following Eq. (6.33) independently of wave numbers $\boldsymbol{k}$.

The final amplitude of each mode depends sensitively on the parameters. As the shear parameter increases, the final amplitude of $\tilde{b}_y$ tends to be larger as is also shown by Chagelishvili *et al.* [52], while the mixing effect on $\tilde{b}_x$ increases. It should be noted that the instability drive G asymptotically has the effect to shift the phase of the oscillations as seen in Eqs. (6.36) and (6.37). However, it does not affect the principal time dependence. The combined effect of the Alfvén wave propagation and shear flow mixing always overcomes the interchange drive. The oscillation of the magnetic flux asymptotically decays with proportionality to the inverse power of time.

6.6 Interchange perturbations perpendicular to ambient magnetic field

When the wave vector is purely perpendicular to the ambient magnetic field, the formulation using the flux function (6.23) fails. As for the condition with $k_{\parallel} = 0$, we discuss the evolution of the stream function ϕ, where $k_{\parallel}$ is a parallel wave number to the ambient magnetic field. The governing equations are Eqs. (6.17) and (6.18), since the flux freezing equation can be decoupled due to the fact that $\boldsymbol{B}_0 \cdot \nabla = 0$. Applying $\partial_t + v_y\partial_y$ to both sides of Eq. (6.17) and substituting it into Eq. (6.18), we obtain

$$(\partial_t + v_y\partial_y)^2\Delta\phi = -\frac{\rho_0' g}{\rho_0}\partial_y^2\phi, \tag{6.38}$$

for a case of linear shear flow. We represent ϕ in terms of the shearing mode given in Eq. (6.25),

$$\phi(\boldsymbol{x}, t) = \int \tilde{\phi}_k(t)\,\tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x})\,\mathrm{d}\boldsymbol{k}. \tag{6.39}$$

By substituting Eq. (6.39) into Eq. (6.38), the following ODE is obtained

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\left(\tilde{k}_x(t)^2 + k_y^2\right)\tilde{\phi}\right] = k_y^2\gamma_G^2\tilde{\phi}, \tag{6.40}$$

where $\gamma_G^2 = -\rho_0' g/\rho_0$ $(= \tau_G^{-2})$ denotes the characteristic growth rate of the interchange instability. Here again we have dropped the subscript k for the sake of simplicity. In order to investigate the time asymptotic behavior of each mode, we assume $t \gg k_x/k_y\sigma$ and $t \gg 1/\sigma$. Then Eq. (6.40) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\tilde{\phi} + \frac{4}{t}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\phi} + \frac{2-\alpha}{t^2}\tilde{\phi} = 0, \tag{6.41}$$

where $\alpha = \gamma_G^2/\sigma^2$ denotes the ratio between the interchange destabilizing effect and flow shear stabilizing one (Richardson number). Note that this ODE is not dependent on the wave numbers $\boldsymbol{k}$. The general solution of Eq. (6.41) is

$$\tilde{\phi} = C_1 t^{m_+} + C_2 t^{m_-}, \tag{6.42}$$

where

$$m_{\pm} = \frac{-3 \pm \sqrt{1 + 4\alpha}}{2}. \tag{6.43}$$

The time asymptotic behavior is therefore determined by the larger index m_+. Thus we can state the condition for the boundedness of $\tilde{\phi}$ as

$$\alpha \le 2 \quad \Rightarrow \quad -\frac{1}{2}\frac{\rho_0' g}{\rho_0} \le \sigma^2. \tag{6.44}$$

The condition for the boundedness of $\tilde{\phi}$ is improved compared with the static case $(\rho_0' \ge 0)$ due to the mixing effect of the shear flow. It is noted that this interchange

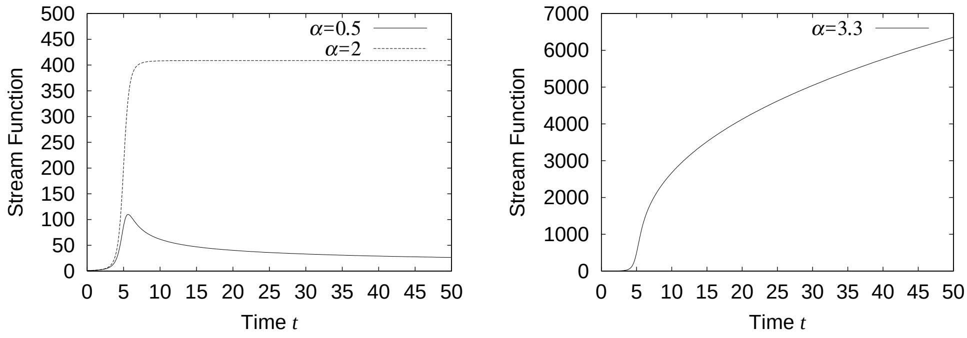

Figure 6.4: Numerical integrations of Eq. (6.40) for different α. The parameters are follows: $k_x = 10$, $k_y = 2$, $\sigma = 1$ and initial perturbations $\tilde{\phi} = 0$ and $\mathrm{d}\tilde{\phi}/\mathrm{d}t = 1.0$ at $t = 0$. The amplitude of the stream function in case of $\alpha = 3.3$ shows the algebraic growth corresponding to $m_+ \simeq 0.35$.

instability can be linearly unstable while the case with $k_\parallel \neq 0$ is completely stabilized. The numerical integrations of the ODE (6.40) are illustrated in Fig. 6.4. The transient behavior is observed until $t_* = 5$, and the asymptotic behavior follows. The asymptotic behavior is algebraic with the power of m_+ as analytically pointed out.

We notice that the 'stability condition' is not well defined here. If we impose the boundedness of $\tilde{v}_y = \mathrm{i}\tilde{k}_x(t)\tilde{\phi} \sim t^{1+m_+}$, the same condition $\rho_0' \geq 0$ as the static case is obtained. However, if we consider the boundedness of other fields which are represented by higher derivatives, e.g. the vortex perturbation, more strict condition will be required. Since the mixing effect of the shear flow distorts the structure of the perturbations into smaller scales, the fields characterized by the higher derivatives will have stronger secularities. Unlike the static case where the evolution of the perturbations can be expressed in the common exponential form, different quantities exhibit different time evolutions in shear flow systems. This could be a pathological problem of defining the 'stability condition' for shear flow systems.

6.7 Effect of magnetic shear on sheared plasma flow

In order to consider the effect of the magnetic shear on sheared plasma flow, let us consider the original linearized MHD equations instead of stream and flux functions, which can be written in the Cartesian coordinates as

$$\rho_0\Big(\partial_t \boldsymbol{v}_1 + \boldsymbol{v}_0 \cdot \nabla \boldsymbol{v}_1 + v_{1x}\partial_x \boldsymbol{v}_0\Big) = \frac{\boldsymbol{B}_0 \cdot \nabla \boldsymbol{b}}{\mu_0} - \nabla\Big(p_0 + \frac{\boldsymbol{B}_0 \cdot \boldsymbol{b}}{\mu_0}\Big), \tag{6.45}$$

$$\partial_t \boldsymbol{b} + \boldsymbol{v}_0 \cdot \nabla \boldsymbol{b} = \boldsymbol{B}_0 \cdot \nabla \boldsymbol{v}_1 + b_x \partial_x \boldsymbol{v}_0, \tag{6.46}$$

where $\boldsymbol{b}$ denotes the perturbed magnetic field and $\boldsymbol{v}_0 = (0, \sigma x, 0)$. Assuming $\boldsymbol{B}_0 = (0, B_{0y}(x), B_{0z}(x))$, we can transform the coordinate as $(x, y, z) \mapsto (x, \eta, \zeta)$ with ζ along the local ambient magnetic field line and η perpendicular to x and ζ. In this coordinates, we have the stationary flow expressed as $(0, v_{0\eta}(x), v_{0\zeta}(x))$. Here, the spatial dependence of the velocity components is,

$$\begin{aligned} v_{0\eta} &= \frac{1}{B_0} B_{0z} \sigma x, \\ v_{0\zeta} &= \frac{1}{B_0} B_{0y} \sigma x. \end{aligned} \tag{6.47}$$

If the magnetic field is homogeneous, the coordinate transformation is also spatially homogeneous and these velocity components are still linear functions with respect to x. Writing the above MHD equations in the new coordinates yields

$$\rho(\partial_t u + v_{0\eta}\partial_\eta u + v_{0\zeta}\partial_\zeta u) = \frac{B_0 \partial_\zeta b_x}{\mu_0} - \partial_x \Big(p + \frac{B_0 b_\zeta}{\mu_0}\Big), \tag{6.48}$$

$$\rho\Big(\partial_t v + v_{0\eta}\partial_\eta v + v_{0\zeta}\partial_\zeta v + \frac{B_{0z}}{B_0}\sigma u\Big) = \frac{B_0 \partial_\zeta b_\eta}{\mu_0} - \partial_\eta \Big(p + \frac{B_0 b_\zeta}{\mu_0}\Big), \tag{6.49}$$

$$\rho\Big(\partial_t w + v_{0\eta}\partial_\eta w + +v_{0\zeta}\partial_\zeta w + \frac{B_{0y}}{B_0}\sigma u\Big) = \frac{B_0 \partial_\zeta b_\zeta}{\mu_0} - \partial_\zeta \Big(p + \frac{B_0 b_\zeta}{\mu_0}\Big), \tag{6.50}$$

$$\partial_t b_x + v_{0\eta}\partial_\eta b_x + v_{0\zeta}\partial_\zeta b_x = B_0 \partial_\zeta u, \tag{6.51}$$

$$\partial_t b_\eta + v_{0\eta}\partial_\eta b_\eta + v_{0\zeta}\partial_\zeta b_\eta = B_0 \partial_\zeta v + \frac{B_{0z}}{B_0}\sigma b_x, \tag{6.52}$$

$$\partial_t b_\zeta + v_{0\eta}\partial_\eta b_\zeta + v_{0\zeta}\partial_\zeta b_\zeta = B_0 \partial_\zeta w + \frac{B_{0y}}{B_0}\sigma b_x, \tag{6.53}$$

where u, v, and w denote the x, η, and ζ components of the perturbed velocity, respectively. Here the evolution of the amplitude $\tilde{b}_x$ is governed by the same equation as Eq. (6.30) for $\tilde{\psi}$.

If we include the magnetic shear, the inhomogeneity is also introduced in the above coordinate transformation. As is seen from Eqs. (6.47), it brings about a nonlinear spatial dependence of the background shear flow profile even if it is assumed to be linear in original Cartesian coordinates. Thus, it is considered that introduction of the magnetic shear seems equivalent to the study of the shear flow different from the linear dependence on x.

6.8 Summary

Kelvin's method of shearing modes is interpreted as a combination of both modal and characteristic methods for the analysis of a non-Hermitian system. The shear flow distorts each Fourier mode, resulting in change of the wave number, which represents the stretching effect of the shear flow (see Fig. 6.1). It is noted that the solution obtained by this method gives the general solution of the system due to the completeness of the sinusoidal function in the Hilbert space.

By means of this method, we have first analyzed the incompressible electromagnetic perturbations in the presence of an interchange drive and obtained the ordinary differential equation (6.30) for the amplitude of the modes $\tilde{\psi}_k$. All modes show an asymptotic decay proportional to the inverse power of time (non-exponential) without any threshold value. This means that the interchange instabilities are always damped away at sufficiently large time due to the combined effect of the Alfvén wave propagation and distortion of modes by means of the background shear flow; i.e. the phase mixing effect. However, the transient behavior is not common for all modes, which depends on the initial wave numbers. Some of them show transient amplifications which are even faster than the interchange mode in the static case. These amplifications are so conspicuous that they may lead to the break down of the linearity of the perturbations.

It should be noted that, since our treatment considers the case of parallel linear shear flow, Kelvin-Helmholtz instabilities, which originate from the second order spatial derivative of the background shear flow [7, 9], are beyond the scope of the present theory. From a mathematical point of view, we stress that the Kelvin-Helmholtz instability is a problem involving purely non-Hermitian operators in the sense that the operator $\mathcal{A}$ of Eq. (6.1) itself becomes non-Hermitian. Thus, the method developed in Sec. 6.2 cannot be applied. This is a well known instability in fluid mechanics whose rigorous mathematical treatment includes highly non-trivial difficulties. We will try to construct a spectral theory on this problem in Chap. 7.

We note that the ODE which gives the evolution of the amplitudes of the interchange modes (6.30) and that of kink-type modes (Eq. (32) in Ref. [143]) are mathematically equivalent. Of course these two modes may have spatially different structures at least for static equilibria. However, these modes have no difference in time evolution by means of our treatment. Thus, we can say that these terms have the same effect in the sense that they enlarge the spectrum to unstable eigenvalues. The equivalence stems from the assumption of a spatially homogeneous magnetic field. However, as discussed in Sec. 6.7, the inhomogeneity of the magnetic field hinders the applicability of Kelvin's method.

We have also investigated the time evolution for purely perpendicular perturbations ($\boldsymbol{k} \cdot \boldsymbol{B}_0 = 0$), which do not excite the Alfvén wave, since they do not bend the magnetic field line during their growth. The flow shear has been shown to have a stabilizing effect also on purely perpendicular disturbances; however, the phase mixing effect alone cannot completely stabilize the interchange instabilities. The condition for the boundedness of the mode amplitudes $\tilde{\phi}_k$ can be expressed in Eq. (6.44) by means of a ratio of instability drive to shear parameter of the mean flow. We have shown that the time evolution of these unstable modes is again of algebraic type. Notice that the conditions for the boundedness of different quantities do not coincide. The discrepancies originate from the fact that, in shear flow systems, different fields experience algebraic evolutions characterized by different powers of time, while the time evolutions for any fields are expressed in a common exponential form for static systems.