Chapter 7

Spectral theory for surface wave model of Rayleigh equation

7.1 Introduction

A variety of complex phenomena occur in non-Hermitian dynamical systems. Non-orthogonality of eigenfunctions (modes) is the deterministic characteristic of such systems, which brings about interactions among different modes. This aspect resembles the mode couplings in nonlinear systems, and hence, the diversity of transient behavior in non-Hermitian systems is rather rich [143, 131, 84].

Let us consider an autonomous evolution equation of the Schrödinger type

$$\begin{cases} \mathrm{i}\partial_t u = \mathcal{H}u \\ u(0) = u_0 \end{cases}, \tag{7.1}$$

where $\mathcal{H}$ is a certain linear operator. When we can generate an exponential function (propagator) $e^{-\mathrm{i}t\mathcal{H}}$, we can write the solution of Eq. (7.1) as

$$u(t) = e^{-\mathrm{i}t\mathcal{H}}u_0.$$

When $u \in \mathbb{C}$ and $\mathcal{H} \in \mathbb{C}$, then $e^{-\mathrm{i}t\mathcal{H}}$ is nothing but the exponential function of elementary mathematics. For vectors $u \in \mathbb{C}^N$ and a linear map $\mathcal{H} : \mathbb{C}^N \to \mathbb{C}^N$, we can define $e^{-\mathrm{i}t\mathcal{H}}$ by the standard power series

$$e^{-\mathrm{i}t\mathcal{H}} = \sum_{n=1}^{\infty} \frac{(-\mathrm{i}t\mathcal{H})^n}{n!}, \tag{7.2}$$

or the Cauchy integral (inverse Laplace transform)

$$e^{-\mathrm{i}t\mathcal{H}} = \frac{1}{2\pi\mathrm{i}} \oint e^{-\mathrm{i}t\lambda}(\lambda\mathcal{I} - \mathcal{H})^{-1}\, \mathrm{d}\lambda, \tag{7.3}$$

where $\mathcal{I}$ denotes the identity operator. For u in a Hilbert space V, $\mathcal{H}$ is an operator in V. For some different classes of operators, we have theories to generate $e^{-\mathrm{i}t\mathcal{H}}$ [28]. A most general theory of generating an exponential function of the type $e^{t\mathcal{A}}$ for positive t (so-called semigroup theory) may be the one due to Hille and Yosida. Although this theory warrants the solvability of initial value problems for a wide class of generators, understanding of the behavior of the solution is not simple. Indeed, the exponential functions of matrices or operators are not necessarily 'exponential' in the conventional sense.

The von Neumann theory for Hermitian (selfadjoint) operators provides a deep insight into the structure of $e^{-\mathrm{i}t\mathcal{H}}$, which invokes the spectral resolution of the generator $\mathcal{H}$ in terms of a complete set of orthogonal modes. The basic idea is that the $e^{-\mathrm{i}t\mathcal{H}}$ may be represented as a sum of independent harmonic oscillators, each of which is an eigenfunction of $\mathcal{H}$ and the corresponding eigenvalue (real number) gives the frequency of the oscillation. Unlike the case of finite dimensional vector space, however, the conventional eigenfunctions may not be complete to span the Hilbert space. The most essential generalization needed to study the infinite dimensional space was the introduction of continuous spectra that correspond to singular eigenfunctions. The spectral resolution of $\mathcal{H}$ is, in general, given by an integral over the spectra (an example will be given in Sec. 7.3). The contribution to the $e^{-\mathrm{i}t\mathcal{H}}$ from the continuous spectra brings about the 'phase mixing' of oscillations with continuous frequencies, resulting in various types of damping. Hence, the reality of the spectra of an Hermitian operator does not necessarily imply a simple sum of harmonic oscillations.

For a linear map in a finite dimensional vector space, the spectral resolution yields the Jordan canonical form, and the explicit representation of $e^{-\mathrm{i}t\mathcal{H}}$ can be constructed using the canonical form. In a Hilbert space, however, such a general theory of spectral resolution is limited to either compact operators or Hermitian ones [11, 6]. This chapter is an attempt to obtain a spectral resolution of a non-Hermitian operator that is not included in the above mentioned categories. The operator is related to an important physics problem (e.g. Kelvin-Helmholtz instability in neutral fluids [83, 138, 7]).

Before formulating the equation, we highlight the essential characteristic of non-Hermitian operator by revisiting the spectral resolution in a finite dimensional vector space. When $\mathcal{H}$ is not a normal map, we can transform it into the Jordan canonical form. By a regular map P, we can transform

$$P^{-1}\mathcal{H}P = \mathcal{J}_1 \dot{+} \mathcal{J}_2 \dot{+} \cdots \dot{+} \mathcal{J}_\nu,$$

where $\dot{+}$ denotes the direct sum of linear maps, and each $\mathcal{J}_j$ is the Jordan block corresponding to the eigenvalue λ_j of $\mathcal{H}$ [$(\lambda_j\mathcal{I} - \mathcal{J}_j)$ is a nilpotent of order N_j, i.e.,

$(\lambda_j I - \mathcal{J}_j)^{N_j} = 0]$, which is represented by the Jordan matrix of order N_j:

$$\mathcal{J}_j = \begin{pmatrix} \lambda_j & 1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ & \ddots & \ddots & 1 \\ 0 & & 0 & \lambda_j \end{pmatrix}. \tag{7.4}$$

When $\mathcal{H}$ is a normal map, all N_j are unity. Then, $\mathcal{H}$ can be diagonalized, and all modes (eigenvectors) are decoupled. A Jordan block of order ≥ 2 represents 'unremovable' interactions among modes. It is remarkable that the canonical representation (7.4) shows such interactions in the form of one-by-one couplings.

Writing

$$e^{-\mathrm{i}t\mathcal{H}} = e^{-\mathrm{i}t\lambda_j} e^{\mathrm{i}t(\lambda_j \mathcal{I} - \mathcal{H})} = e^{-\mathrm{i}t\lambda_j} \left[\mathcal{I} + \mathrm{i}t(\lambda_j \mathcal{I} - \mathcal{H}) - \frac{t^2(\lambda_j \mathcal{I} - \mathcal{H})^2}{2} + \cdots \right],$$

we find that the $e^{-\mathrm{i}t\mathcal{H}}$ acting on the (generalized) eigenspace belonging to λ_j includes factors

$$e^{-\mathrm{i}t\lambda_j}, \; te^{-\mathrm{i}t\lambda_j}, \; \cdots, \; t^{N_j-1}e^{-\mathrm{i}t\lambda_j}.$$

Therefore, even if every eigenvalue λ_j is real, the $e^{-\mathrm{i}t\mathcal{H}}$ can describe an 'instability' (growth of oscillation amplitude). The algebraic growth of amplitudes (the factors t^p) is called 'secularity.'

On the stability analysis of a fluid, we mainly have two methods which are widely used in literatures. One is based on variational problem [65, 78, 142], and the other is based on the spectral method [37, 98]. Actually, necessary and sufficient condition of Rayleigh equation to have complex eigenvalues is shown in Ref. [37] by means of Nyquist criterion, however, the completeness of the solution is still left open. We will focus on the completeness of the spectral solution and investigate the possibility of the secular behavior. We will introduce a generalized Rayleigh equation in Sec. 7.2, and discuss the properties of two consisting operators in Sec. 7.3. By introducing surface wave model [134, 73, 7], we can describe the system rather simply. Firstly, we will formally investigate the possibility of resonance among modes in Secs. 7.4 and 7.5. Spectral resolution of the generator is given in Sec. 7.6, which will be compared with the Laplace transform method in Sec. 7.7. In Sec. 7.8, we will summarize the obtained results.

7.2 Generalized Rayleigh equation

The vortex dynamics equation in $\mathbb{R}^2$ [the coordinates are denoted by (x, y)] reads as a Liouville equation

$$\partial_t \Psi + \{H, \Psi\} = 0, \tag{7.5}$$

where Ψ is the vorticity, H is the Hamiltonian (stream function) of an incompressible flow $\boldsymbol{v} = (\partial_y H, -\partial_x H)^t$ that transports the vortices, and

$$\{a, b\} = (\partial_y a)(\partial_x b) - (\partial_x a)(\partial_y b) = -\nabla a \times \nabla b \cdot \nabla z$$

is the Poisson bracket.

When the Hamiltonian H depends on Ψ, the evolution equation (7.5) is nonlinear. The dynamics of Ψ can couple with other fields when they are included in H. The simplest example of nonlinear vortex dynamics is the Euler fluid (incompressible ideal flow), where

$$-\Delta H = \Psi, \tag{7.6}$$

or, denoting the Green operator of the Laplacian $-\Delta$ by $\mathcal{K}$

$$H = \mathcal{K}\Psi. \tag{7.7}$$

Physical examples of relevant phenomena are rather rich; the Rossby waves of perturbations in geological jet streams, the diocotron waves in non-neutral plasmas, and the drift waves in magnetized plasmas.

Let us linearize Eq. (7.5) with decomposing Ψ and H into their ambient (denoted by subscript 0) and fluctuation (denoted by subscript 1) parts:

$$\begin{aligned} \Psi &= \Psi_0 + \Psi_1, \\ H &= H_0 + H_1 = \mathcal{K}\Psi_0 + \mathcal{K}\Psi_1. \end{aligned}$$

By neglecting the second-order terms, Eq. (7.5) leads to

$$\partial_t \Psi_1 + \{H_0, \Psi_1\} + \{\Delta H_0, \mathcal{K}\Psi_1\} = 0. \tag{7.8}$$

Hereafter we will omit the subscript 1 denoting the perturbed field for simplicity.

In this chapter, we consider one-dimensional problem with

$$H_0 = H_0(x).$$

Since the ambient Hamiltonian H_0 is independent of y, the wavenumber in y becomes a good quantum number and we can replace ∂_y by $\mathrm{i}k$. We write

$$v(x) = -\partial_x H_0(x),$$

to obtain the standard Rayleigh equation

$$\mathrm{i}\partial_t \Psi = kv(x)\Psi + kv''(x)\mathcal{K}\Psi. \tag{7.9}$$

The Green operator $\mathcal{K}$ is represented by a convolution integral

$$(\mathcal{K}f)(x) = \int_{-\infty}^{+\infty} \frac{e^{-k|x-\xi|}}{2k} f(\xi)\,\mathrm{d}\xi. \tag{7.10}$$

In the following, we denote the Green function by $K(x,\xi)$;

$$K(x,\xi) = \frac{e^{-k|x-\xi|}}{2k}. \tag{7.11}$$

Here we will define a generalized Rayleigh equation as

$$\mathrm{i}\partial_t \Psi = \mathcal{L}\Psi \tag{7.12}$$

$$\mathcal{L} = kv(x) + kw(x)\mathcal{K}, \tag{7.13}$$

where $v(x)$ and $w(x)$ can be independent arbitrary functions. It corresponds to the generalization of the Rayleigh equation (7.9). The case when $w(x) = v''(x)$ recovers the physically relevant equation (7.9) where $v(x)$ denotes the steady flow velocity in the y direction.

7.3 Formal spectra of generalized Rayleigh equation

The generator of the vortex dynamics equation (7.12) consists of two terms, each of which describes different mechanism of the vortex motion. The first term on the right-hand side of Eq. (7.12) [originating from $\{H_0, \Psi\}$ in Eq. (7.8)] represents the transport of the vorticity by the ambient flow $v(x)$. An inhomogeneous (sheared) flow distorts vortices, and hence, no stationary structure can persist in a shear flow [$v(x) \neq$ const]. Such a dynamics is described by a continuous spectrum. On the other hand, the second term [originating from $\{\Delta H_0, \mathcal{K}\Psi\}$ in Eq. (7.8)] describes the interaction between the perturbation and the ambient field. When the ambient vorticity $\Psi_0 = -\Delta H_0$ has a spatial gradient, a flow induced by a perturbation yields a local change of the vorticity. This term, hence, can create perturbed vortices from the ambient field.

Firstly, let us assume $w(x) = 0$ in Eq. (7.13) and consider

$$\mathrm{i}\partial_t \Psi = kv(x)\Psi \tag{7.14}$$

with a 'continuous' real function $v(x)$, which reads as a Schrödinger equation with a Hamiltonian $kv(x)$.

The formal eigenvalue and the corresponding eigenfunction of the generator of Eq. (7.14), with setting

$$kv(x)\Psi = \lambda\Psi$$

[i.e., $\Psi(t) = e^{-\mathrm{i}\lambda t}\Psi$], is given by

$$\lambda = kv(\mu), \quad \Psi = \delta(x-\mu), \tag{7.15}$$

where μ is an arbitrary real number and δ denotes the delta-measure. For convenience, we write

$$(\delta(x-\mu), f(x)) = \int_{-\infty}^{+\infty} \delta(x-\mu) f(x)\,\mathrm{d}x = f(\mu).$$

A formal spectral resolution of the generator is written as

$$\begin{aligned} kv(x)f(x) &= \int_{-\infty}^{+\infty} kv(\mu)(\delta(x-\mu), f)\,\delta(x-\mu)\,\mathrm{d}\mu \\ &= \int_{-\infty}^{+\infty} kv(\mu)f(\mu)\delta(x-\mu)\,\mathrm{d}\mu. \end{aligned} \tag{7.16}$$

Rigorous mathematical representation of this 'continuous spectrum' is given by the spectral resolution of the coordinate operator:

$$xf(x) = \int_{-\infty}^{+\infty} \mu\,\mathrm{d}E(\mu)f(x), \tag{7.17}$$

where $\{E(\mu);\ \mu \in \mathbb{R}\}$ is a family of projectors defined by

$$E(\mu)f(x) = \begin{cases} f(x) & \text{for } x \le \mu \\ 0 & \text{for } x > \mu \end{cases}. \tag{7.18}$$

The projector $E(\mu)$ gives a resolution of the identity:

$$\mathcal{I} = \int_{-\infty}^{+\infty} \mathrm{d}E(\mu). \tag{7.19}$$

Using this representation of the coordinate operator, we can write

$$kv(x)f(x) = \int_{-\infty}^{+\infty} kv(\mu)\,\mathrm{d}E(\mu)f(x), \tag{7.20}$$

which represents the spectral resolution of the generator $kv(x)$ in terms of its generalized eigenfunctions [this corresponds to the continuous version of Eq. (B.28)]. The solution of Eq. (7.14) with initial condition $\Psi(x,0)$ is given by

$$\Psi(x,t) = \int_{-\infty}^{+\infty} e^{-\mathrm{i}tkv(\mu)}\,\mathrm{d}E(\mu)\Psi(x,0) = e^{-\mathrm{i}tkv(x)}\Psi(x,0). \tag{7.21}$$

Next, let us assume

$$v(x) = 0, \tag{7.22}$$

$$w(x) = -\frac{U}{a}[\delta(x-a) - \delta(x+a)], \tag{7.23}$$

then the operator can be written as

$$\mathcal{L}_1 = -\frac{U}{2a}[\delta(x-a) - \delta(x+a)] \int_{-\infty}^{\infty} e^{-k|x-\xi|} \ \cdot \ \mathrm{d}\xi \tag{7.24}$$

There are two solutions which match with the exponential time evolution. Assuming $\varphi \propto e^{-\mathrm{i}\lambda t}$ and substituting it in the place of Ψ, then we obtain the eigenvalue problem

$$\lambda\varphi(x) = -\frac{U}{2a}[\delta(x-a) - \delta(x+a)] \int_{-\infty}^{\infty} e^{-k|x-\xi|}\varphi(\xi)\,\mathrm{d}\xi. \tag{7.25}$$

The eigenvalues and the corresponding eigenfunctions are

$$\lambda = \pm\frac{U}{2a}\sqrt{1-e^{-4ka}}, \quad \varphi(x) = \delta(x-a) - \left(1 \pm \sqrt{1-e^{-4ka}}\right) e^{2ka}\,\delta(x+a). \tag{7.26}$$

Therefore we can write the operator $\mathcal{L}$ in the form of the matrix by taking the basis vector as $\delta(x-a)$ and $\delta(x+a)$,

$$\mathcal{L}_1 = \frac{U}{2a}\begin{pmatrix} -1 & -e^{-2ka} \\ e^{-2ka} & 1 \end{pmatrix}. \tag{7.27}$$

These oscillation denote the coupled diocotron modes which are excited on both surfaces $x = \pm a$. Note that these modes are stable for positive k in the case of no continuous spectra. Since the governing equation is same as the case of diocotron instabilities in non-neutral plasmas [60, 91], these oscillations are called 'diocotron oscillation.'

If we take $v(x)$ to be consistent with $w(x)$ of Eq. (7.23) as

$$v(x) = \begin{cases} -U & (x \le -a) \\ Ux/a & (-a < x < a) \\ U & (a \le x) \end{cases}, \tag{7.28}$$

and again focusing on the subspace spanned by two surface waves, then the operator $\mathcal{L}_2$ is expressed in the matrix form as

$$\mathcal{L}_2 = \begin{pmatrix} kU & 0 \\ 0 & -kU \end{pmatrix} + \mathcal{L}_1. \tag{7.29}$$

This 2×2 matrix $\mathcal{L}_2$ can be readily diagonalized in terms of the non-unitary transform and thus it turns out to be a semi-simple type supposed that $k \neq 1/2a$. Its two eigenvalues denote the famous Kelvin-Helmholtz instability [7], whose dispersion relation is written as

$$\lambda^2 = \frac{U^2}{4a^2}\left[(1-2ka)^2 - e^{-4ka}\right]. \tag{7.30}$$

If λ degenerates to zero, we may have secularity, and moreover, the other part of the matrix may include Jordan block. These details will be discussed later.

7.4 Resonance between point and continuous spectra

We have studied the spectra of the operators separately in the previous section. However, we have to be careful about the resonance (frequency overlapping) between modes. Namely the eigenvalues Eqs. (7.15) and (7.26) may overlap. In this section, we will formally show the effect coming from such resonance or frequency overlapping.

Let us first consider the following consistent case where the operator is expressed as

$$\mathcal{L}_3 = kv(x) - \frac{U}{2a}\delta(x-a)\int_{-\infty}^{\infty} e^{-k|x-\xi|}\ \cdot\ \mathrm{d}\xi, \tag{7.31}$$

$$v(x) = \begin{cases} Ux/a & (x < a) \\ U & (a \le x) \end{cases}. \tag{7.32}$$

By taking the component proportional to $\delta(x-a)$ separately, we divide the vorticity as

$$\varPsi(x,t) = \alpha(t)\delta(x-a)\ +\ \tilde{\varPsi}(x,t) \tag{7.33}$$

where $\tilde{\varPsi}$ does not include any singularity on $x = a$. Then, we can divide the time evolution equation as

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t) = \frac{U}{2a}(2ka-1)\alpha(t) - \frac{U}{2a}\int_{-\infty}^{\infty} e^{-k|a-\xi|}\tilde{\varPsi}(\xi,t)\,\mathrm{d}\xi, \tag{7.34}$$

$$\mathrm{i}\partial_t\tilde{\varPsi}(x,t) = kv(x)\tilde{\varPsi}(x,t). \tag{7.35}$$

The second equation can be readily integrated for each x, which gives

$$\tilde{\varPsi}(x,t) = e^{-\mathrm{i}tkv(x)}\tilde{\varPsi}(x,0). \tag{7.36}$$

Plugging it into Eq. (7.34), we have

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t) = \frac{U}{2a}(2ka-1)\alpha(t) - \frac{U}{2a}\int_{-\infty}^{\infty} e^{-k|a-\xi|}e^{-\mathrm{i}tkv(\xi)}\tilde{\varPsi}(\xi,0)\,\mathrm{d}\xi, \tag{7.37}$$

where the last integration denotes the phase mixing of each singular eigenfunction. It can be integrated to give

$$\alpha(t) = e^{-\mathrm{i}\omega_1 t}\alpha(0) + \frac{U}{2a}\int e^{-k|a-\xi|}\frac{e^{-\mathrm{i}\omega_1 t} - e^{-\mathrm{i}tkv(\xi)}}{kv(\xi)-\omega_1}\tilde{\varPsi}(\xi,0)\,\mathrm{d}\xi, \tag{7.38}$$

where $\omega_1 = (2ka-1)U/2a$ represents the Doppler shifted diocotron frequency.

By assuming $\varphi \propto e^{-\mathrm{i}\lambda t}$, we will obtain the following eigenvalue problem.

$$\lambda\varphi(x) = kv(x)\varphi(x) - \frac{U}{2a}\delta(x-a)\int_{-\infty}^{\infty} e^{-k|x-\xi|}\varphi(\xi)\,\mathrm{d}\xi. \tag{7.39}$$

For this eigenvalue problem, we have the following sets of eigenvalues and the corresponding eigenfunctions:

1. For $\lambda_0 = kU$; the corresponding eigenfunctions are arbitrary continuous functions $\varphi_0(x)$ which satisfy

$$\int_a^\infty e^{k(a-\xi)}\varphi_0(\xi)\,\mathrm{d}\xi = 0, \quad \wedge \quad \varphi_0(x) = 0\ (x < a). \tag{7.40}$$

2. For $\lambda_1 = kU - U/2a$; the corresponding eigenfunction is

$$\varphi_1(x) = \delta(x-a). \tag{7.41}$$

3. For $\lambda_\mu = kU\mu/a$ ($\mu < a \ \wedge\ \mu \neq a - 1/2k$); the corresponding eigenfunctions are

$$\varphi_\mu(x) = \delta(x-\mu) + \frac{e^{-k(a-\mu)}}{2k(a-\mu)-1}\delta(x-a). \tag{7.42}$$

However, these eigenvalues are not complete and we have another eigenfunction in a wider sense. That is $\varphi_2(x) = \delta(x-\mu_0)$ where $\mu_0 = a - 1/2k$. We can easily see that

$$(\lambda_1 - \mathcal{L}_3)\varphi_2(x) = \frac{U}{2a}e^{-k(a-\mu_0)}\varphi_1(x), \tag{7.43}$$

where $\lambda_1 = kU - U/2a$. Of course $(\lambda_1 - \mathcal{L}_3)^2\varphi_2(x) = 0$ also holds. These relations are quite similar to the eigenfunction in a wider sense for the finite dimensional matrix operator. Thus we may be able to write the operator $\mathcal{L}_3$ in the following matrix form including the case $\mu = \mu_0$:

$$\mathcal{L}_3 = \begin{pmatrix} kU - \dfrac{U}{2a} & -\dfrac{U}{2a}e^{-k(a-\mu)} \\ 0 & kU\dfrac{\mu}{a} \end{pmatrix}, \tag{7.44}$$

where a Jordan block is obtained when $kU - U/2a = kU\mu/a$ ($\mu = \mu_0$).

Let us evaluate the time evolution of the perturbation when we have taken this $\varphi_2(x)$ as an initial condition. As we can see from Eq. (7.43), we will have $\varphi_1(x)$ component by applying the generator $\mathcal{L}_3$ on the initial condition $\varphi_2(x)$. Here we may be able to consider the evolution to be closed in the functional space spanned by $\varphi_1(x)$ and $\varphi_2(x)$. Thus it may be natural to assume Ψ as

$$\Psi(x,t) = \sum_{i=1}^{2}\alpha_i(t)\,\varphi_i(x). \tag{7.45}$$

Substituting this expression into the original equation (7.12), we obtain

$$\begin{aligned}
\mathrm{i}\partial_t(\alpha_1\varphi_1+\alpha_2\varphi_2) &= \mathcal{L}_3(\alpha_1\varphi_1+\alpha_2\varphi_2)\\
&= \lambda_1\alpha_1\varphi_1+\alpha_2\mathcal{L}_3\varphi_2\\
&= \Big(\lambda_1\alpha_1-\frac{U}{2a\sqrt{e}}\alpha_2\Big)\varphi_1+\lambda_1\varphi_2 \qquad (7.46)
\end{aligned}$$

When we decompose Eq. (7.46) into φ_1 and φ_2 by considering them to be independent, we obtain two coupled time evolution equations;

$$\frac{\mathrm{d}\alpha_1}{\mathrm{d}t} = -\mathrm{i}\lambda_1\alpha_1+\mathrm{i}\frac{U}{2a\sqrt{e}}\alpha_2, \qquad (7.47)$$

$$\frac{\mathrm{d}\alpha_2}{\mathrm{d}t} = -\mathrm{i}\lambda_1\alpha_2. \qquad (7.48)$$

The latter one can be readily integrated to give

$$\alpha_2(t) = \alpha_2(0)e^{-\mathrm{i}\lambda_1 t}. \qquad (7.49)$$

The former equation can also be readily solved with the variable constant method.

$$\alpha_1(t) = \Big[\mathrm{i}\frac{U}{2a\sqrt{e}}\alpha_2(0)t+\alpha_1(0)\Big]e^{-\mathrm{i}\lambda_1 t}. \qquad (7.50)$$

Here we see that for $\alpha_2(0)\neq 0$, we may have a secularity due to the resonance of the diocotron mode with one of the singular eigenfunction in the continuous spectrum. It is noted that even if all eigenvalues are real, we may have instability due to its secular evolution. However, by introducing a proper Hilbert space, we may show that this apparent secularity is virtual. Physically, it is considered that the algebraic growth of the surface wave does not occur when the resonance is the form that the wave energy flows *from inner singular function to surface wave.*

7.5 Kelvin-Helmholtz system

Let us consider the case with two singular breaks which may lead to the well known Kelvin-Helmholtz instabilities. The generator is written as

$$\mathcal{L}_4 = kv(x)-\frac{U}{2a}[\delta(x-a)-\delta(x+a)]\int_{-\infty}^{\infty} e^{-k|x-\xi|}\ \cdot\ \mathrm{d}\xi, \qquad (7.51)$$

where the velocity field is defined as

$$v(x) = \begin{cases} -U & (x\le -a)\\ Ux/a & (-a<x<a)\\ U & (a\le x)\end{cases}. \qquad (7.52)$$

If we put the basis vectors as $\varphi_1(x) = \delta(x-a)$, $\varphi_\mu(x) = \delta(x-\mu)$, and $\varphi_3(x) = \delta(x+a)$, then we can obtain the following matrix representation for the operator $\mathcal{L}_4$;

$$\mathcal{L}_4 = \frac{U}{2a}\begin{pmatrix} 2ka-1 & -e^{-k(a-\mu)} & -e^{-2ka} \\ 0 & 2k\mu & 0 \\ e^{-2ka} & e^{-k(a+\mu)} & -(2ka-1) \end{pmatrix} \tag{7.53}$$

Note here that $\mu = \pm\mu_0$ does not create a Jordan block any more since the frequencies of diocotron oscillations are shifted due to their coupling.[1]

If we expand the perturbed vortex field as

$$\Psi(x,t) = \sum_{i=1}^{3} \alpha_i(t)\,\varphi_i(x), \tag{7.54}$$

and substitute it into the Rayleigh equation, we obtain

$$\mathrm{i}\frac{\mathrm{d}\alpha_1}{\mathrm{d}t} = \frac{U}{2a}\left[(2ka-1)\alpha_1 - e^{-k(a-\mu)}\alpha_2 - e^{-2ka}\alpha_3\right], \tag{7.55}$$

$$\mathrm{i}\frac{\mathrm{d}\alpha_2}{\mathrm{d}t} = kU\frac{\mu}{a}\alpha_2, \tag{7.56}$$

$$\mathrm{i}\frac{\mathrm{d}\alpha_3}{\mathrm{d}t} = \frac{U}{2a}\left[e^{-2ka}\alpha_1 + e^{-k(a+\mu)}\alpha_2 - (2ka-1)\alpha_3\right]. \tag{7.57}$$

The coupling between two diocotron oscillations is eliminated by diagonalizing the matrix of 2×2 part of the outermost. The eigenvalues are found to be

$$\lambda_1^\dagger = \frac{U}{2a}e^{-2ka}\sinh\psi, \tag{7.58}$$

$$\lambda_3^\dagger = -\frac{U}{2a}e^{-2ka}\sinh\psi, \tag{7.59}$$

and the corresponding eigenfunctions are

$$\varphi_1^\dagger(x) = \delta(x-a) + e^{-\psi}\delta(x+a), \tag{7.60}$$

$$\varphi_3^\dagger(x) = e^{-\psi}\delta(x-a) + \delta(x+a), \tag{7.61}$$

[1] It may be easily understood by considering a simple example

$$\mathcal{A} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 3 & 0 \\ -1 & -2 & -3 \end{pmatrix}.$$

This matrix can be readily diagonalized and the eigenvalues become

$$\lambda = 3,\ \pm 2\sqrt{2},$$

which are no longer degenerated, and the corresponding eigenfunctions are

$$\varphi = \left(1, \frac{1}{10}, -\frac{1}{5}\right),\ (1, 0, -3 \pm 2\sqrt{2}).$$

where $\cosh\psi = (2ka-1)e^{2ka}$. Thus we define new coefficients which denote the amplitudes of $\varphi_1^\dagger$ and $\varphi_2^\dagger$ as

$$\alpha_1\varphi_1 + \alpha_3\varphi_3 = \beta_1\varphi_1^\dagger + \beta_3\varphi_3^\dagger. \tag{7.62}$$

From this relation we obtain

$$\begin{pmatrix} \beta_1 \\ \beta_3 \end{pmatrix} = \frac{2}{\sinh\psi}\begin{pmatrix} e^\psi & -1 \\ -1 & e^\psi \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_3 \end{pmatrix}. \tag{7.63}$$

With these new coefficients, we can decouple the diocotron modes, and obtain the following three equations;

$$\mathrm{i}\frac{\mathrm{d}\beta_1}{\mathrm{d}t} = \lambda_1^\dagger\beta_1 - \frac{U}{a\sinh\psi}[e^{-k(a-\mu)}e^\psi + e^{-k(a+\mu)}]\alpha_2, \tag{7.64}$$

$$\mathrm{i}\frac{\mathrm{d}\alpha_2}{\mathrm{d}t} = kU\frac{\mu}{a}\alpha_2, \tag{7.65}$$

$$\mathrm{i}\frac{\mathrm{d}\beta_3}{\mathrm{d}t} = \lambda_3^\dagger\beta_3 + \frac{U}{a\sinh\psi}[e^{-k(a-\mu)} + e^{-k(a+\mu)}e^\psi]\alpha_2. \tag{7.66}$$

The solution of Eq. (7.65) is given as

$$\alpha_2(t) = \alpha_2(0)\, e^{-\mathrm{i}(kU\mu/a)t}. \tag{7.67}$$

If we assume

$$kU\frac{\mu}{a} = \lambda_1^\dagger \ \text{ or } \ \lambda_3^\dagger, \tag{7.68}$$

then we might have secularity due to the resonance with the simple oscillator $\varphi_2^\dagger$. On the other hand, we *do not* have it, when the eigenvalues $\lambda_{1,3}^\dagger$ are complex or pure imaginary, namely when the system is unstable in a Kelvin-Helmholtz sense.

7.6 Spectral resolution of coupled non-Hermitian generator

In this section, we formulate the vortex dynamics equation (7.12) for a case with a sum of the delta-measure field

$$w(x) = \sum_{j=1}^{N} A_j\delta(x-a_j) \quad (A_j, a_j \in \mathbb{R},\ j=1,\ldots,N), \tag{7.69}$$

as an evolution equation in an appropriate Hilbert space, and give a spectral resolution of the generator. The generator reads

$$\begin{aligned}\mathcal{L}\varPsi &= kv(x)\varPsi + kw(x)\mathcal{K}\varPsi \\ &= kv(x)\varPsi + \sum_{j=1}^{N} kA_j\delta(x-a_j)\int_{-\infty}^{+\infty} K(x,\xi)\varPsi(\xi)\, d\xi,\end{aligned} \tag{7.70}$$

where $kv(x) \in C(\mathbb{R})$, $A_j \in \mathbb{R}$, $a_j \in \mathbb{R}$ $(j = 1, \ldots, N)$, and $K(x, \xi) = e^{-k|x-\xi|}/2k$ is the Green function [see Eq. (7.11)]. In what follows, we assume

$$|v(x)| < c \quad (\forall x)$$

with a finite number c.

It is noted that, since the delta measure $\delta(x - a_j)$ is not a member of the Lebesgue space, we encounter a difficulty in formulating the problem in the conventional L^2 Hilbert space.

7.6.1 Mathematical formulation of the generator

Let us consider a Hilbert space

$$V = \mathbb{C}^N \oplus L^2(\mathbb{R}), \tag{7.71}$$

where $\mathbb{C}^N$ is the unitary space of dimension N, and $L^2(\mathbb{R})$ is the complex Lebesgue space on $\mathbb{R}$ endowed with the standard inner product. The member of V is written as

$$\varPsi = \begin{pmatrix} \boldsymbol{\alpha} \\ \tilde{\varPsi}(x) \end{pmatrix} \quad [\boldsymbol{\alpha} \in \mathbb{C}^N,\ \tilde{\varPsi}(x) \in L^2(\mathbb{R})]. \tag{7.72}$$

The inner product of V is, thus, defined as

$$\begin{aligned} (\varPsi \mid \varPsi^\dagger) &= (\boldsymbol{\alpha}, \boldsymbol{\alpha}^\dagger) + (\tilde{\varPsi}, \tilde{\varPsi}^\dagger) \\ &= \sum_{j=1}^{N} \bar{\alpha}_j \alpha_j^\dagger + \int_{-\infty}^{+\infty} \bar{\tilde{\varPsi}}(x) \tilde{\varPsi}^\dagger(x)\, \mathrm{d}x \end{aligned} \tag{7.73}$$

We identify

$$\varPsi = \begin{pmatrix} \boldsymbol{\alpha} \\ \tilde{\varPsi}(x) \end{pmatrix} \ \Leftrightarrow \ \varPsi(x) = \sum_{j=1}^{N} \alpha_j \delta(x - a_j) \ + \ \tilde{\varPsi}(x). \tag{7.74}$$

It is essential to decompose the delta-measure part (representing the surface waves) from the total vorticity $\varPsi$. Although the supports (in the sense of distributions) of both components $\delta(x - a_j)$ and $\tilde{\varPsi}(x)$ may overlap, we separate them into different degrees of freedom. Because $\mathcal{K}\varPsi \in C(\mathbb{R})$ for all $\varPsi \in V$, the generator $\mathcal{L}$ is a bounded operator on V.

Following Eq. (7.74), the generator $\mathcal{L}$ of Eq. (7.70) is now written in a matrix form

$$\mathcal{L}\varPsi = \begin{pmatrix} \omega_1(a_1) & \cdots & kA_1K(a_1, a_N) & \int kA_1K(a_1, x) \cdot\ \mathrm{d}x \\ \vdots & \ddots & \vdots & \vdots \\ kA_NK(a_N, a_1) & \cdots & \omega_N(a_N) & \int kA_NK(a_N, x) \cdot\ \mathrm{d}x \\ 0 & \cdots & 0 & \omega_{\mathrm{c}}(x) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \\ \tilde{\varPsi}(x) \end{pmatrix}, \tag{7.75}$$

where we have introduced the notation

$$\omega_j(a_j) = kv(a_j) + \frac{A_j}{2} \quad (j = 1, \ldots, N), \tag{7.76}$$

denoting the Doppler shifted diocotron frequency, and $\omega_{\rm c}(x) = kv(x)$, respectively.

In the previous section, we dealt delta functions in a formal way and did calculations with $\delta(x-\mu)$ for an arbitrary $\mu \in \mathbb{R}$ [see Eq. (7.15)]. We note that such formal functions are not the member of the Hilbert space V. In this section, however, they are justified as generalized eigenfunctions corresponding to 'continuous spectra.'

7.6.2 Spectral resolution of the generator

First, we consider the simple case of single 'source,' i.e., $w(x) = A\delta(x-a)$ (see Sec. 7.4). The surface wave mode has only one degree of freedom ($N = 1$). Here, the generator $\mathcal{L}$ of Eq. (7.75) simplifies as

$$\mathcal{L} = \begin{pmatrix} \omega_1(a) & \int kAK(a,x)\cdot\ dx \\ 0 & \omega_{\rm c}(x) \end{pmatrix}. \tag{7.77}$$

As we have shown in Sec. 7.4, there are mainly two different classes of formal eigenfunctions [see Eq. (7.41) and Eq. (7.42)]. According to the notation of Eq. (7.72), the eigenvalues and the corresponding eigenfunctions are shown as

$$\omega_1(a) = kv(a) + \frac{A}{2}, \quad \boldsymbol{U}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7.78}$$

$$\omega_{\rm c}(\mu) = kv(\mu), \quad \tilde{\boldsymbol{U}}_{\rm c}(\mu) = \begin{pmatrix} \frac{m(\mu)kAK(a,\mu)}{\omega_{\rm c}(\mu)-\omega_1(a)} \\ \tilde{m}(\mu)\delta(x-\mu) \end{pmatrix}, \tag{7.79}$$

where $\tilde{m}$ is defined in order to unify both the non-resonant and resonant (nilpotent) cases as

$$\tilde{m}(\mu) = \begin{cases} m(\mu) & \text{if } \omega_{\rm c}(\mu) \neq \omega_1(a) \\ (kAK(a,\mu))^{-1} & \text{if } \omega_{\rm c}(\mu) = \omega_1(a) \text{ (i.e. } m(\mu) = 0), \end{cases} \tag{7.80}$$

with the normalizer

$$m(\mu) = \left[1 + \left(\frac{kAK(a,\mu)}{\omega_{\rm c}(\mu) - \omega_1(a)} \right)^2 \right]^{-1/2}. \tag{7.81}$$

The first eigenfunction (7.78) represents the surface wave. The second one (7.79) includes an arbitrary real number μ, corresponding to the continuous spectrum, and a singular function $\delta(x-\mu)$. We must integrate Eq. (7.79) over $\mu \in \mathbb{R}$ to span the complete basis of V. Formally, we can define the non-unitary transform

$$\begin{aligned}\mathcal{T} &= \left(\boldsymbol{U}_1 \quad \int (\delta(x-\mu),\ \cdot\ )\tilde{\boldsymbol{U}}_c(\mu)\,\mathrm{d}\mu\right) \\ &= \begin{pmatrix} 1 & \int (\delta(x-\mu),\ \cdot\ )\frac{m(\mu)AK(a,\mu)}{\omega_c(\mu)-\omega_1(a)}\,\mathrm{d}\mu \\ 0 & \int (\delta(x-\mu),\ \cdot\ )\tilde{m}(\mu)\delta(x-\mu)\,\mathrm{d}\mu \end{pmatrix}. \end{aligned} \tag{7.82}$$

To cast this formal expression in an appropriate mathematical representation, we invoke the resolution of the identity (7.19). The formal correspondence is

$$\int_{-\infty}^{+\infty} (\delta(x-\mu), u(x))\,\delta(x-\mu)\,\mathrm{d}\mu = \int_{-\infty}^{+\infty} \mathrm{d}E(\mu)u = u.$$

We also define

$$F(\mu)u = \int_{-\infty}^{\mu} u(x)\,\mathrm{d}x, \tag{7.83}$$

which gives

$$\mathrm{d}F(\mu)u = u(\mu)\,\mathrm{d}\mu.$$

With this notation, we can write

$$\int f(\mu)\,\mathrm{d}F(\mu)u(x) = \int f(\mu)u(\mu)\,\mathrm{d}\mu = \int f(x)u(x)\,\mathrm{d}x.$$

The operator $\mathcal{T}$ is now written in a rigorous form as

$$\mathcal{T} = \begin{pmatrix} 1 & \int \frac{m(\mu)kAK(a,\mu)}{\omega_c(\mu)-\omega_1(a)}\mathrm{d}F(\mu) \\ 0 & \int \tilde{m}(\mu)\,\mathrm{d}E(\mu) \end{pmatrix} = \begin{pmatrix} 1 & \int \frac{m(x)kAK(a,x)}{\omega_c(x)-\omega_1(a)}\cdot\ \mathrm{d}x \\ 0 & \tilde{m}(x) \end{pmatrix} \tag{7.84}$$

Reflecting the non-Hermitian property of the generator $\mathcal{L}$, the operator $\mathcal{T}$ is not a unitary transform. By combing both non-resonant and resonant (nilpotent) cases, this $\mathcal{T}$ gives a regular transform. The inverse operator is

$$\mathcal{T}^{-1} = \begin{pmatrix} 1 & -\int \left(\frac{m(x)}{\tilde{m}(x)}\right)\left(\frac{kAK(a,x)}{\omega_c(x)-\omega_1(a)}\right)\cdot\ \mathrm{d}x \\ 0 & \tilde{m}(x)^{-1} \end{pmatrix}. \tag{7.85}$$

With the transforms $\mathcal{T}$ and $\mathcal{T}^{-1}$, we obtain the Jordan canonical form of $\mathcal{L}$;

$$\begin{aligned}\mathcal{T}^{-1}\mathcal{L}\mathcal{T} &= \begin{pmatrix} \omega_1 & \int \rho(\mu)\,\mathrm{d}F(\mu) \\ 0 & \int \omega_c(\mu)\,\mathrm{d}E(\mu) \end{pmatrix} \\ &= \begin{pmatrix} \omega_1 & \int \rho(x)\cdot\ \mathrm{d}x \\ 0 & \omega_c(x) \end{pmatrix}, \end{aligned} \tag{7.86}$$

where

$$\rho(x) = \begin{cases} 1 & \text{if } \omega_c(\mu) = \omega_1(a) \\ 0 & \text{if } \omega_c(\mu) \neq \omega_1(a) \end{cases}.$$

The support of $\rho(x)$ may have a finite measure when the resonance condition $\omega_c(\mu) = \omega_1(a)$ holds on a finite interval of x.

7.6.3 Spectral representation of the propagator

The propagator $e^{-\mathrm{i}t\mathcal{L}}$ is defined by solving the initial value problem for Eq. (7.12)

$$\begin{cases} \mathrm{i}\partial_t \Psi = \mathcal{L}\Psi \\ \Psi(0) = \Psi^0 \end{cases}, \tag{7.87}$$

and writing the solution as

$$\Psi(t) = e^{-\mathrm{i}t\mathcal{L}}\Psi^0.$$

By introducing $\Psi = \mathcal{T}\chi$, Eq. (7.87) is transformed into

$$\begin{cases} \mathrm{i}\partial_t \chi = \mathcal{T}^{-1}\mathcal{L}\mathcal{T}\chi \\ \chi(0) = \mathcal{T}^{-1}\Psi^0 \end{cases}. \tag{7.88}$$

Using the spectral resolution (7.86), the solution of Eq. (7.88) is given by

$$\begin{aligned} e^{-\mathrm{i}t\mathcal{T}^{-1}\mathcal{L}\mathcal{T}} &= \begin{pmatrix} e^{-\mathrm{i}t\omega_1} & -\int \mathrm{i}te^{-\mathrm{i}t\omega_1}\rho(\mu)\,\mathrm{d}F(\mu) \\ 0 & \int e^{-\mathrm{i}t\omega_\mathrm{c}(\mu)}\,\mathrm{d}E(\mu) \end{pmatrix} \\ &= \begin{pmatrix} e^{-\mathrm{i}t\omega_1} & -\int \mathrm{i}te^{-\mathrm{i}t\omega_1}\rho(x)\cdot\ \mathrm{d}x \\ 0 & e^{-\mathrm{i}t\omega_\mathrm{c}(x)} \end{pmatrix}. \end{aligned} \tag{7.89}$$

The solution of Eq. (7.87) is given by

$$\Psi(t) = \mathcal{T}\left[e^{-it\mathcal{T}^{-1}\mathcal{L}\mathcal{T}}\right]\mathcal{T}^{-1}\Psi^0.$$

With Eqs. (7.84) and (7.85), we obtain

$$\begin{aligned} e^{-\mathrm{i}t\mathcal{L}} &= \mathcal{T}\begin{pmatrix} e^{-\mathrm{i}t\omega_1} & -\int \mathrm{i}te^{-\mathrm{i}t\omega_1}\rho(x)\cdot\ \mathrm{d}x \\ 0 & e^{-\mathrm{i}t\omega_\mathrm{c}(x)} \end{pmatrix}\mathcal{T}^{-1} \\ &= \begin{pmatrix} e^{-\mathrm{i}t\omega_1} & X \\ 0 & e^{-\mathrm{i}t\omega_\mathrm{c}(x)} \end{pmatrix}, \end{aligned} \tag{7.90}$$

where

$$X = \int \left([1-\rho(x)] \frac{[e^{-\mathrm{i}t\omega_\mathrm{c}(x)} - e^{-\mathrm{i}t\omega_1(a)}]kAK(a,x)}{\omega_\mathrm{c}(x) - \omega_1(a)} - \mathrm{i}te^{-\mathrm{i}t\omega_1}kAK(a,x)\rho(x) \right) \cdot\ \mathrm{d}x,$$

and we have used the relations

$$\begin{cases} \dfrac{m(x)}{\tilde{m}(x)} = 1 - \rho(x) \\ \dfrac{\rho(x)}{\tilde{m}(x)} = kAK(a,x)\rho(x) \end{cases}.$$

In the case of multiple sources [see Eq. (7.69)], the first class of formal eigenfunctions are obtained by solving

$$\begin{pmatrix} \omega_1(a_1) & \cdots & kA_1K(a_1,a_N) \\ \vdots & \ddots & \vdots \\ kA_NK(a_N,a_1) & \cdots & \omega_N(a_N) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} = \omega \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix}, \tag{7.91}$$

and the second class by

$$\begin{pmatrix} \omega_1(a_1)-\omega_{\rm c}(\mu) & \cdots & kA_1K(a_1,a_N) \\ \vdots & \ddots & \vdots \\ kA_NK(a_N,a_1) & \cdots & \omega_N(a_N)-\omega_{\rm c}(\mu) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} = -\beta \begin{pmatrix} kA_1K(a_1,\mu) \\ \vdots \\ kA_NK(a_N,\mu) \end{pmatrix}. \tag{7.92}$$

Then we can define the transform as

$$\mathcal{T} = \left(\boldsymbol{U}_1 \ \cdots \ \boldsymbol{U}_N \ \int (\delta(x-\mu), \ \cdot \ )\tilde{\boldsymbol{U}}_c(\mu)\, \mathrm{d}\mu \right), \tag{7.93}$$

where $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_N$ denote the formal eigenfunctions of the first class, respectively, and $\tilde{\boldsymbol{U}}_c$ denote those of second class which are obtained in the way shown above. Again by combining both non-resonant and resonant (nilpotent) cases, this transform $\mathcal{T}$ is not unitary but regular. Thus, the inverse operator $\mathcal{T}^{-1}$ can be defined and $e^{-\mathrm{i}t\mathcal{T}^{-1}\mathcal{L}\mathcal{T}}$ can be generated.

7.7 Laplace transformation

In this section, we will solve the same initial value problem by means of Laplace transformation which is another method to give a general solution for bounded operators. The Laplace transformation is defined here by

$$\hat{f}(x,s) = \int_0^\infty f(x,t)\, e^{-st}\, \mathrm{d}t, \tag{7.94}$$

where s satisfies the condition that its real part is larger than any temporal singularity of the function $f(x,t)$ for the convergence of the integration. The inversion will be given by

$$f(x,t) = \frac{1}{2\pi \mathrm{i}} \int_{s_0-\mathrm{i}\infty}^{s_0+\mathrm{i}\infty} \hat{f}(x,s)\, e^{st}\, \mathrm{d}s, \tag{7.95}$$

where $s_0 = \mathrm{Re}(s) > 0$. Due to the definition of our Hilbert space (7.71), perturbed vorticity will be transformed as

$$\hat{\Psi}(x,s) = \begin{pmatrix} \hat{\boldsymbol{\alpha}}(s) \\ \hat{\tilde{\Psi}}(x,s) \end{pmatrix} = \begin{pmatrix} \displaystyle\int_0^\infty \boldsymbol{\alpha}(t)\, e^{-st}\, \mathrm{d}t \\ \\ \displaystyle\int_0^\infty \tilde{\Psi}(x,t)\, e^{-st}\, \mathrm{d}t \end{pmatrix}. \tag{7.96}$$

The generalized Rayleigh equation (7.12) will be transformed by multiplying e^{-st} on both sides and integrating with respect to time as

$$[\mathrm{i}s - v(x)]\hat{\Psi} - w(x)\mathcal{K}\hat{\Psi} = i\Psi(0), \tag{7.97}$$

where the integrations with respect to ξ (in the operator $\mathcal{K}$) and t has been commuted.

Let us consider at first the single source case. Plugging

$$w(x) = A\delta(x-a) \tag{7.98}$$

and Eq. (7.96) into Eq. (7.97), we have

$$[s + \mathrm{i}\omega_1(a)]\hat{\alpha}(s) + \mathrm{i}\int_{-\infty}^{\infty} kAK(a,\xi)\hat{\tilde{\Psi}}(\xi,s)\,d\xi = \alpha(0), \tag{7.99}$$

$$[s + \mathrm{i}\omega_{\mathrm{c}}(x)]\hat{\tilde{\Psi}}(x,s) = \tilde{\Psi}(x,0), \tag{7.100}$$

or

$$\begin{pmatrix} s + \mathrm{i}\omega_1(a) & \mathrm{i}\int kAK(a,\xi)\cdot\ \mathrm{d}\xi \\ 0 & s + \mathrm{i}\omega_{\mathrm{c}}(x) \end{pmatrix} \begin{pmatrix} \hat{\alpha} \\ \hat{\tilde{\Psi}} \end{pmatrix} = \begin{pmatrix} \alpha(0) \\ \tilde{\Psi}(0) \end{pmatrix}, \tag{7.101}$$

in the matrix form. Equation (7.100) for the inner vorticity fluctuation will be readily solved for each x as

$$\hat{\tilde{\Psi}}(x,s) = \frac{\tilde{\Psi}(x,0)}{s + \mathrm{i}\omega_{\mathrm{c}}(x)}, \tag{7.102}$$

and its pole at $s = -\mathrm{i}\omega_{\mathrm{c}}(x)$ will give a simple oscillation

$$\tilde{\Psi}(x,t) = \tilde{\Psi}(x,0)\,e^{-\mathrm{i}t\omega_{\mathrm{c}}(x)}, \tag{7.103}$$

which exactly coincides with the result obtained by Case [50]. It is shown for the Couette flow that the continuous spectrum exhibits the phase mixing damping of the velocity field v_{1x} proportional to $1/t$ when integrated.

Equation (7.102) will be substituted into Eq. (7.99), which now reads as

$$\hat{\alpha}(s) = \frac{\alpha(0)}{s + \mathrm{i}\omega_1(a)} - \frac{\mathrm{i}}{s + \mathrm{i}\omega_1(a)}\int_{-\infty}^{+\infty} \frac{kAK(a,\xi)\,\tilde{\Psi}(\xi,0)}{s + \mathrm{i}\omega_{\mathrm{c}}(\xi)}\,\mathrm{d}\xi, \tag{7.104}$$

where the isolated pole in the first term gives again the simple oscillation when inverted. By inverting the Laplace transformation, we formally obtain

$$\alpha(t) = \alpha(0)\,e^{-\mathrm{i}t\omega_1(a)} - \frac{1}{2\pi}\int_{s_0-\mathrm{i}\infty}^{s_0+\mathrm{i}\infty} \frac{e^{st}}{s + \mathrm{i}\omega_1(a)}\int_{-\infty}^{+\infty} \frac{kAK(a,\xi)\,\tilde{\Psi}(\xi,0)}{s + \mathrm{i}\omega_{\mathrm{c}}(\xi)}\,\mathrm{d}\xi\,\mathrm{d}s. \tag{7.105}$$

The second term has two singularities which apparently looks same as the case of kinetic treatment for electrostatic oscillations (see Appendix C). However, since the numerator of the integrand is not an analytic function in all region on the real axis of ξ, the analytic continuation cannot be properly defined. Thus, we have to choose another method here. Fortunately, double integrations in the second term are commutable since both of them are uniformly converging. Thus, the integration with respect to s can be evaluated by deforming the integration path as

$$\begin{aligned}
&\frac{1}{2\pi}\int_{s_0-\mathrm{i}\infty}^{s_0+\mathrm{i}\infty}\frac{e^{st}}{s+\mathrm{i}\omega_1(a)}\int_{-\infty}^{+\infty}\frac{kAK(a,\xi)\,\tilde{\Psi}(\xi,0)}{s+\mathrm{i}\omega_{\mathrm{c}}(\xi)}\,\mathrm{d}\xi\,\mathrm{d}s\\
&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}\xi\,AK(a,\xi)\,\tilde{\Psi}(\xi,0)\int_{s_0-\mathrm{i}\infty}^{s_0+\mathrm{i}\infty}\frac{e^{st}}{[s+\mathrm{i}\omega_1(a)][s+\mathrm{i}\omega_{\mathrm{c}}(\xi)]}\,\mathrm{d}s\\
&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}\xi\,\frac{kAK(a,\xi)\,\tilde{\Psi}(\xi,0)}{\mathrm{i}\omega_{\mathrm{c}}(\xi)-\mathrm{i}\omega_1(a)}\int_{s_0-\mathrm{i}\infty}^{s_0+\mathrm{i}\infty}\left(\frac{1}{s+\mathrm{i}\omega_1(a)}-\frac{1}{s+\mathrm{i}\omega_{\mathrm{c}}(\xi)}\right)e^{st}\,\mathrm{d}s\\
&=\int_{-\infty}^{+\infty}\frac{[e^{-\mathrm{i}t\omega_1(a)}-e^{-\mathrm{i}t\omega_{\mathrm{c}}(\xi)}]kAK(a,\xi)}{\omega_{\mathrm{c}}(\xi)-\omega_1(a)}\,\tilde{\Psi}(\xi,0)\,\mathrm{d}\xi. \qquad (7.106)
\end{aligned}$$

Then, the final expression for the surface wave evolution is given by

$$\alpha(t)=\alpha(0)\,e^{-\mathrm{i}t\omega_1(a)}+\int_{-\infty}^{+\infty}\frac{[e^{-\mathrm{i}t\omega_1(a)}-e^{-\mathrm{i}t\omega_{\mathrm{c}}(\xi)}]kAK(a,\xi)}{\omega_1(a)-\omega_{\mathrm{c}}(\xi)}\,\tilde{\Psi}(\xi,0)\,\mathrm{d}\xi. \qquad (7.107)$$

By combining Eq. (7.103) with Eq. (7.107), the solution for the initial value problem is expressed in the following matrix form;

$$\begin{pmatrix}\alpha(t)\\ \tilde{\Psi}(x,t)\end{pmatrix}=\begin{pmatrix}e^{-\mathrm{i}t\omega_1(a)} & \int_{-\infty}^{+\infty}\frac{[e^{-\mathrm{i}t\omega_1(a)}-e^{-\mathrm{i}t\omega_{\mathrm{c}}(x)}]kAK(a,x)}{\omega_1(a)-\omega_{\mathrm{c}}(x)}\cdot\,\mathrm{d}x\\ 0 & e^{-\mathrm{i}t\omega_{\mathrm{c}}(x)}\end{pmatrix}\begin{pmatrix}\alpha(0)\\ \tilde{\Psi}(x,0)\end{pmatrix}, \qquad (7.108)$$

which exactly coincides with the previous expression (7.90) obtained by means of the spectral method. It is noted that the value of the function

$$\frac{[e^{-\mathrm{i}t\omega_1(a)}-e^{-\mathrm{i}t\omega_{\mathrm{c}}(x)}]}{\omega_1(a)-\omega_{\mathrm{c}}(x)} \qquad (7.109)$$

at zeros of the denominator should be defined by the value $-\mathrm{i}te^{-\mathrm{i}t\omega_1(a)}$ of the limit $\omega_{\mathrm{c}}(x)\to\omega_1(a)$.

7.8 Summary

We have obtained the spectral resolution of the non-Hermitian operator for the surface wave model of Kelvin-Helmholtz instability. With the aid of dividing the Hilbert space into a finite discrete part and an infinite continuous part, we have shown that the operator of Rayleigh equation is bounded. It is found that the system has a resonance (frequency overlapping) between the inner vorticity fluctuation and the surface wave. When the resonance occurs in a finite measure region which introduces the point spectra in the inner vorticity fluctuation, the resonance gives the energy transfer from the inner vorticity fluctuation to the surface wave. This leads to the secular behavior of the surface wave. However, when the resonance occurs in a zero measure region, the energy cannot be transferred from the continuous spectrum to the point one. Therefore, the surface wave just represents the asymptotic oscillation.

In the case of kinetic treatment for electrostatic oscillations [95], the multiplication is given by a simple coordinate operator which only contains the continuous spectrum. Therefore, the resonance happens to balance with the phase mixing due to continuum damping, and leads to the stationary asymptotic behavior. Here, the situation of Kelvin-Helmholtz instability is quite similar to the electrostatic oscillations, but the multiplication is given by a function. Thus, the Kelvin-Helmholtz instability may bring about the point spectra and secular behavior of the surface wave. In the physical situation, however, the resonance with finite measure may not be rarely satisfied. In this case, it may not be considered to produce secularity. By modifying the system, resonance may happen between two continuous spectra. Then, a realistic secular behavior is produced [84].