Resonance between continuous spectra: Secular behavior of Alfvén waves in a flowing plasma

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Conventional normal mode analysis often falls short in predicting a variety of transient phenomena in a non-self-adjoint (non-Hermitian) system. Laplace transform is capable of capturing all possible behavior in general systems. However, degenerate essential spectra require careful analysis. The Alfvén wave in a flowing plasma is an example in which the coalescence of the Alfvén singularities yields nonexponential growth of fluctuations. Invoking hyperfunction theory, rigorous expression of the Laplace transform leads to an accurate estimate of the asymptotic behavior of resonant singular modes. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1834591]

I. INTRODUCTION

Waves and instabilities in plasmas are far richer than those observed in fluids, solids, or various continuous media. Dynamical representations of perturbations are beyond the scope of conventional normal mode analysis—a perturbation $u(x,t)$ may not assume the form of $u(x,t)=e^{-i\omega t}\psi(x)$; hence, we cannot replace ∂_t with $-i\omega$ in the determining evolution equations.[1] In analyzing a variety of transient phenomena in plasmas, we encounter two difficult problems that require careful mathematical considerations. One is the general non-self-adjoint (non-Hermitian) property of plasmas. When plural branches of waves overlap in some ranges of frequencies, these waves may interact through resonance (because the modes are not *orthogonal*), resulting in algebraic (secular) amplification of the wave. The other is the existence of various continuous spectra (essential singularities in the dispersion relation). Resonance between continuous spectra is not as simple as those in point spectra (eigenvalues), because we must analyze singular eigenfunctions (see Ref. 2 and Appendix A).

As is well known, Laplace transform is capable of capturing all possible behavior in general systems. However, the inverse Laplace transform (equivalent to the Dunford integral of the spectral theory) is not easy when multiple continuous spectra are degenerated. In the present paper, we invoke the hyperfunction theory[3,4] to provide a rigorous basis for dealing with singular eigenfunctions (see Appendix B). The theory reveals a natural relation between the Fourier transform (characterizing the eigenfunctions) and the Laplace transform (defining the solution of an initial value problem). We can derive an accurate estimate of the asymptotic behavior of resonant singular modes, which was left out in earlier theories.[5–7]

The subject of our analysis is the Alfvén waves governed by the ideal magnetohydrodynamics (MHD) equations. Linearizing the MHD equations around an equilibrium with velocity $\mathbf{V}$, magnetic field $\mathbf{B}$, and pressure P, we obtain an evolution equation for the fluctuation parts $\tilde{\mathbf{v}}$, $\tilde{\mathbf{b}}$, and $\tilde{p}$,

$$i\partial_t f = \mathcal{K} f, \quad f = {}^t(\tilde{\mathbf{v}}\tilde{\mathbf{b}}\tilde{p}), \tag{1}$$

where the generator $\mathcal{K}$ is a linear differential operator. We will solve the initial value problem of (1) by assuming incompressibility.

Note that the non-self-adjointness discussed here is not peculiar to flowing plasmas, since the well-known self-adjoint property of static plasmas is attributed to the Lagrange representation. Introducing the Lagrange variable $\tilde{\boldsymbol{\xi}}$ as $\tilde{\mathbf{v}}=\partial_t\tilde{\boldsymbol{\xi}}+\mathbf{V}\cdot\nabla\tilde{\boldsymbol{\xi}}-\tilde{\boldsymbol{\xi}}\cdot\nabla\mathbf{V}$, (1) is reduced to

$$\partial_t^2\tilde{\boldsymbol{\xi}} + 2\mathbf{V}\cdot\nabla\partial_t\tilde{\boldsymbol{\xi}} = \mathcal{F}\tilde{\boldsymbol{\xi}}, \tag{2}$$

where $\mathcal{F}$ is a self-adjoint operator under appropriate boundary conditions.[8,9] If the equilibrium is not flowing ($\mathbf{V}\equiv 0$), the evolution of $\tilde{\boldsymbol{\xi}}$ is generated by only $\mathcal{F}$. Due to Von Neumann's theorem of the spectral resolution of the self-adjoint operator,[10] we can invoke the normal mode (spectral) analysis of $\mathcal{F}$ to generate the solution of (2). The MHD stability analysis is, therefore, conventionally based on the dispersion relation where time derivative ($i\partial_t$) is replaced by eigenvalue (ω).

However, in deriving (2), we have assumed *homogeneous* initial conditions which satisfy

$$[\tilde{p} + \tilde{\boldsymbol{\xi}}\cdot\nabla P + \gamma P\nabla\cdot\tilde{\boldsymbol{\xi}}]_{t=0} \equiv 0, \tag{3}$$

$$[\tilde{\mathbf{b}} + \nabla\times(\mathbf{B}\times\tilde{\boldsymbol{\xi}})]_{t=0} \equiv 0, \tag{4}$$

where γ denotes the specific heat ratio. Hence, the solution of (1) is wider than that of (2), and $\mathcal{K}$ is no longer self-adjoint even for static plasmas. In this paper, we will show complicated algebraic growth of fluctuation by solving (1).

In Sec. II, we will formulate the evolution equation including the effect of shear flow. We consider an equilibrium with slab geometry, which is inhomogeneous only in the x direction. Assuming incompressibility, the fluctuations can be represented by four variables $f={}^t(\tilde{w}_x\,\tilde{j}_x\,\tilde{v}_x\,\tilde{b}_x)$. Writing $f_1={}^t(\tilde{w}_x\,\tilde{j}_x)$ and $f_2={}^t(\tilde{v}_x\,\tilde{b}_x)$, the evolution equation can be cast in a block form,

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$$i\partial_t \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \mathcal{K}_1 & \mathcal{N} \\ 0 & \mathcal{K}_2 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{5}$$

where $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{N}$ are linear operators in 2×2 matrix forms. As we will see later, the operator $\mathcal{K}_2$ has only real continuous spectra, $\sigma_c^+=\{\omega_{a+}(x); x\in D\}$ and $\sigma_c^-=\{\omega_{a-}(x); x\in D\}$, representing the (Doppler-shifted) Alfvén waves, where $\omega_{a+}(x)$ and $\omega_{a-}(x)$ denote the local Alfvén frequencies associated with the two propagation directions along magnetic field, and there is no unstable point spectrum under certain conditions proposed by Barston[5] and Stern.[11] If the domain D is bounded, i.e., the thickness of the slab is finite, $\mathcal{K}_2$ is a bounded operator. It will be shown that the operator $\mathcal{K}_1$ is a multiplication operator and has the same spectrum as $\mathcal{K}_2$. Since these four continuous spectra are degenerating and the generator has an off-diagonal element $\mathcal{N}$, we can expect algebraic growth of f_1 by analogy with the Jordan block in linear algebra. However, the nilpotent among continuous spectra is not mathematically resolved, and we therefore must solve the initial value problem.

In Sec. III, the initial value problem will be solved by using the Laplace transform. The major part of this section will be devoted to solving

$$i\partial_t f_2 = \mathcal{K}_2 f_2. \tag{6}$$

Barston,[5] Sedláček,[6] and Tataronis[7] considered the mathematically equivalent problem without ambient flow, where the continuous spectra do not receive the Doppler shift; $\sigma_c^\pm = \{\pm\omega_a(x); x\in D\}$ with $\omega_a(x) := \omega_{a+}(x) = -\omega_{a-}(x)$. They assumed that $\omega_a^2(x) > 0$ for all $x\in D$, which implies that σ_c^+ and σ_c^- are disjoint. In this case, as Sedláček pointed out, the Laplace transform can be translated into the normal modes analysis.

The normal modes analysis does not give a correct solution if there is a point x_r that satisfies $\omega_a(x_r)=0$. In the presence of this x_r (so-called rational surface), the forward and backward Alfvén continuous spectra (σ_c^+ and σ_c^-) overlap. Then we encounter a complicated singularity on the rational surface, which implies a resonance between the two continuous spectra at zero frequency in the subsystem (6). The hyperfunction theory enables analysis of this singularity. As a result, we will find that the solution f_2 asymptotically evolves into a standing wave that yields a magnetic island. This phenomenon was omitted in the earlier works[7,12] where the Alfvén waves (except for zero frequency) receive the phase mixing damping. The standard analysis based on Lagrange representations excludes this solution because of the assumption (4).

The effect of flow becomes essential if the gradient of the flow shear exceeds that of the magnetic shear. Our analysis will prove that the magnetic island cannot survive in such a strong shear flow.

In the end of Sec. III, the equation for f_1 will be solved by using the given f_2. We will observe an algebraic growth localized on the rational surface, which can be understood as a resonance among the four Alfvén continuous spectra. The physical explanation will be discussed in the summary.

II. GENERAL MODEL OF ALFVÉN WAVES

We formulate a system of equations that describes the Alfvén waves in inhomogeneous ambient magnetic field and flow. The incompressible ideal MHD equations read as

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \frac{1}{a^2}\left[-\nabla\frac{|\mathbf{b}|^2}{2} + (\mathbf{b}\cdot\nabla)\mathbf{b}\right], \tag{7}$$

$$\partial_t \mathbf{b} + (\mathbf{v}\cdot\nabla)\mathbf{b} = (\mathbf{b}\cdot\nabla)\mathbf{v}, \tag{8}$$

$$\nabla\cdot\mathbf{v} = 0, \quad \nabla\cdot\mathbf{b} = 0, \tag{9}$$

where $\mathbf{v}$, $\mathbf{b}$, and p denote velocity, magnetic fields, and pressure, respectively. The mass density is assumed to be constant. Variables are in the standard Alfvén units. The scaling parameter a denotes the Alfvén Mach number. In what follows, we choose the representative magnetic field so that $a=1$.

We consider an equilibrium with slab geometry,

$$\mathbf{V}(x) = (0, V_y(x), V_z(x)), \quad \mathbf{B}(x) = (0, B_y(x), B_z(x)),$$
$$\text{and } P(x) + |\mathbf{B}(x)|^2/2 = \text{const}. \tag{10}$$

Since the equilibrium may have strong magnetic shear, we do not invoke the reduced MHD approximation, but consider all components of the vector field in the fluctuation part which depend on all coordinates. The essential independent variable is, however, only x due to the homogeneity of the equilibrium in the y and z directions. We may introduce wave numbers, k_y and k_z, and substitute the following expressions into (7)–(9):

$$\mathbf{v} = \mathbf{V}(x) + \tilde{\mathbf{v}}(x,t)e^{i(k_y y + k_z z)}, \tag{11}$$

$$\mathbf{b} = \mathbf{B}(x) + \tilde{\mathbf{b}}(x,t)e^{i(k_y y + k_z z)}, \tag{12}$$

$$p = P(x) + \tilde{p}(x,t)e^{i(k_y y + k_z z)}. \tag{13}$$

For these fluctuations, the use of so-called *normal velocity* and *normal vorticity* is more convenient (see Appendix C). By introducing the x component of vorticity and current as, respectively,

$$\tilde{w}_x = ik_y\tilde{v}_z - ik_z\tilde{v}_y \quad \text{and} \quad \tilde{j}_x = ik_y\tilde{b}_z - ik_z\tilde{b}_y, \tag{14}$$

the four variables $\tilde{v}_x$, $\tilde{w}_x$, $\tilde{b}_x$, and $\tilde{j}_x$ can reproduce all components of $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{b}}$.

Now we linearize (7)–(9). By taking the divergence of (7), we obtain the linearized Poisson equation

$$\Delta\tilde{p} + \Delta(\mathbf{B}\cdot\tilde{\mathbf{b}}) = -2i\mathbf{k}\cdot\mathbf{V}'\tilde{v}_x + 2i\mathbf{k}\cdot\mathbf{B}'\tilde{b}_x, \tag{15}$$

where $\mathbf{k}=(0,k_y,k_z)$ and a prime ($'$) denotes the x derivative. Using (15), we can eliminate $\tilde{p}$ from the system (i.e., $\tilde{p}$ is not an independent variable). After some manipulations, the linearized equations are written in the matrix form of

$$i\partial_t \begin{pmatrix} \tilde{w}_x \\ \tilde{j}_x \\ \Delta\tilde{v}_x \\ \tilde{b}_x \end{pmatrix} = \begin{pmatrix} \mathbf{k}\cdot\mathbf{V} & -\mathbf{k}\cdot\mathbf{B} & -(\mathbf{V}'\times\mathbf{k})\cdot\mathbf{e}_x\Delta^{-1} & (\mathbf{B}'\times\mathbf{k})\cdot\mathbf{e}_x \\ -\mathbf{k}\cdot\mathbf{B} & \mathbf{k}\cdot\mathbf{V} & -(\mathbf{B}'\times\mathbf{k})\cdot\mathbf{e}_x\Delta^{-1} & (\mathbf{V}'\times\mathbf{k})\cdot\mathbf{e}_x \\ 0 & 0 & \mathbf{k}\cdot\mathbf{V}-\mathbf{k}\cdot\mathbf{V}''\Delta^{-1} & -\mathbf{k}\cdot\mathbf{B}\Delta+\mathbf{k}\cdot\mathbf{B}'' \\ 0 & 0 & -\mathbf{k}\cdot\mathbf{B}\Delta^{-1} & \mathbf{k}\cdot\mathbf{V} \end{pmatrix} \begin{pmatrix} \tilde{w}_x \\ \tilde{j}_x \\ \Delta\tilde{v}_x \\ \tilde{b}_x \end{pmatrix}. \tag{16}$$

where $\Delta=\partial_x^2-k^2$ with $k=|\mathbf{k}|$, and $\mathbf{e}_x=(1,0,0)$ is the unit vector. By considering a domain $D\subset\mathbb{R}$ with a Dirichlet boundary condition, the operator Δ^{-1} is uniquely given as a convolution integral.

In terms of $\mathcal{M}_\pm=\tilde{v}_x\mp\tilde{b}_x$ and $\mathcal{S}_\pm=\tilde{w}_x\mp\tilde{j}_x$, (16) reads as

$$i\partial_t \begin{pmatrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \Delta\mathcal{M}_+ \\ \Delta\mathcal{M}_- \end{pmatrix} = \begin{pmatrix} \omega_{a+} & 0 & 0 & -\iota_-\Delta^{-1} \\ 0 & \omega_{a-} & -\iota_+\Delta^{-1} & 0 \\ 0 & 0 & \omega_{a+}+\omega'_{a+}\partial_x\Delta^{-1} & -\omega'_{a-}\partial_x\Delta^{-1}-\omega''_{a-}\Delta^{-1} \\ 0 & 0 & -\omega'_{a+}\partial_x\Delta^{-1}-\omega''_{a+}\Delta^{-1} & \omega_{a-}+\omega'_{a-}\partial_x\Delta^{-1} \end{pmatrix} \begin{pmatrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \Delta\mathcal{M}_+ \\ \Delta\mathcal{M}_- \end{pmatrix}, \tag{17}$$

where $\omega_{a\pm}(x)=\mathbf{k}\cdot\mathbf{V}(x)\pm\mathbf{k}\cdot\mathbf{B}(x)$ and $\iota_\pm(x)=[\mathbf{V}'(x)\times\mathbf{k}]\cdot\mathbf{e}_x\pm[\mathbf{B}'(x)\times\mathbf{k}]\cdot\mathbf{e}_x$. If the ambient fields are uniform [$\mathbf{V}(x)\equiv$const and $\mathbf{B}(x)\equiv$const], the generator is reduced to a diagonal form, and therefore each variable oscillates independently with the frequency ω_{a+} or ω_{a-}. The variables $\mathcal{M}_\pm$ and $\mathcal{S}_\pm$ correspond to the Alfvén waves polarized in the two directions perpendicular to $\mathbf{B}$, and the subscripts + and − identify the propagation directions.

Inhomogeneity of the ambient fields causes interactions between two polarized waves. The behavior of $\mathcal{M}_\pm$ is not affected by $\mathcal{S}_\pm$, while $\mathcal{S}_\pm$ is forced by $\mathcal{M}_\pm$. In this paper, the lower two equations and the upper two equations in (17) [or (16)] will be referred to as *master* equations and *slave* equations, respectively, and we call $\mathcal{M}_\pm$ (or $\tilde{v}_x$ and $\tilde{b}_x$) *master* variables and $\mathcal{S}_\pm$ (or $\tilde{w}_x$ and $\tilde{j}_x$) *slave* variables.

If we consider normal modes such as $\tilde{v}_x(x,t)=\tilde{v}_x(x)e^{-i\omega t}$, the master equations are combined, by eliminating $\tilde{b}_x$, into

$$\frac{d}{dx}\left[(\omega-\omega_{a+}(x))(\omega-\omega_{a-}(x))\frac{d\tilde{u}}{dx}\right]-k^2(\omega-\omega_{a+}(x))(\omega-\omega_{a-}(x))\tilde{u}=0, \tag{18}$$

where $\tilde{u}=\tilde{v}_x/(\omega-\mathbf{k}\cdot\mathbf{V})$. This Sturmian equation becomes singular if $\omega\in\sigma_c=\sigma_c^+\cup\sigma_c^-$, where σ_c^+ and σ_c^- are defined by $\sigma_c^\pm=\{\omega_{a\pm}(x);x\in D\}$ and correspond to the (Doppler-shifted) Alfvén continuous spectra. For a static equilibrium $[\mathbf{V}(x)\equiv 0]$ and any function $\mathbf{B}(x)$, Barston[5] considered a mathematically equivalent problem to (18) and proved that there is no spectrum in addition to σ_c. Even in the presence of flow, we can prove by applying Stern's result,[11] that exponentially growing or damping mode ($\omega\notin\mathbb{R}$) does not exist if

$$|\mathbf{k}\cdot\mathbf{V}(x)|\leqslant|\mathbf{k}\cdot\mathbf{B}(x)| \tag{19}$$

is satisfied everywhere in a certain inertial frame. Even if (19) is violated, the system might be still free from the exponential growth mode. For example, if both $\mathbf{k}\cdot\mathbf{V}(x)$ and $\mathbf{k}\cdot\mathbf{B}(x)$ are linear functions of x, there is no drive for the Kelvin–Helmholtz instability. We are interested in the continuous spectrum, and therefore consider only stable equilibria in the sense of the dispersion relation.

The slave equations have the same continuous spectra σ_c^+ and σ_c^- due to the multiplication operator $\omega_{a\pm}(x)$ in (17). Therefore, the evolution equation has four degenerate continuous spectra in total. Given that the mathematical structure of (17) is similar to that of Orr–Sommerfeld and Squire equations[13] in fluid dynamics, we can expect the algebraic instability to be caused by the resonant energy transfer from the master equations to the slave one. For example, if we substitute the special configuration

$$\mathbf{V}=(0,0,0),\quad \mathbf{B}=(0,0,x),\quad k_y\neq 0,\quad k_z=0. \tag{20}$$

the generator of (16) is reduced to a number matrix that is called nilpotent in the linear algebra (see Appendix A). Our linear system is, therefore, non-self-adjoint even for static plasmas and the slave variables increase algebraically in proportion to t. This example is too simple because the Alfvén wave does not propagate and the continuous spectra completely degenerate into a point ($\sigma_c=\sigma_c^\pm=\{0\}$). We will discuss a less trivial problem in the next section, where the four Alfvén continuous spectra yield spatially and temporally complicated behavior. Compared with our problem, the Orr–Sommerfeld and Squire equations have only point spectra due to the viscosity, and then the eigenvalue problem is applicable based on some general theories like Ref. 14. In our system, however, we must solve the initial value problem directly because there is no theory for the non-self-adjoint system with degenerate continuous spectra.

Before ending this section, we make a comment on the effect of the compressibility. If we consider a compressible plasma, we must include two other variables into the system, viz., the pressure perturbation ($\tilde{p}$) and the other component of velocity perturbation (say $ik_y\tilde{v}_y+ik_z\tilde{v}_z$). The six variables are generally coupled and no longer decomposed into the master and slave variables as in (17). The continuous spectra that we found in the master and slave equations, then, appear as the slow and Alfvén continuous spectra, respectively. Since the frequency of the slow wave coincides with that of the Alfvén wave on the rational surface, the degeneracy of the four continuous spectra still exists, and hence, the special configuration given by (20) will cause the algebraic growth of $\tilde{v}_z$ even if we take into account the compressibility. Although the similar transient behavior is expected, the analysis of the compressible case is generally involved.

III. ANALYSIS OF ALGEBRAIC BEHAVIOR

In the following analysis, we will solve the initial value problem of (17) by assuming linear profiles of the ambient magnetic field and flow. Let one rational surface exist in the domain and the x coordinate be chosen so that the surface is located on $x=0$. In an appropriate inertial frame, we may assume $\mathbf{k}\cdot\mathbf{V}(0)=0$ without loss of generality. Thus, we have

$$\mathbf{k}\cdot\mathbf{V}(x)=\alpha x,\quad \mathbf{k}\cdot\mathbf{B}(x)=\beta x\quad(\alpha,\beta=\text{const}). \tag{21}$$

In this coordinate system, we consider a finite domain $x \in[-L_1,L_2]$ $(L_1>0,L_2>0)$ with the Dirichlet boundary condition

$$\tilde{v}_x(-L_1,t)=\tilde{v}_x(L_2,t)=0,\quad \tilde{b}_x(-L_1,t)=\tilde{b}_x(L_2,t)=0 \tag{22}$$

for all t. Finally, we assume that the initial conditions are holomorphic functions on $[-L_1,L_2]$. The Alfvén continuous spectra discussed in the preceding section are represented by

$$\sigma_c^+=\{\omega\in\mathbb{R};\omega=(\alpha+\beta)x,x\in[-L_1,L_2]\}, \tag{23}$$

$$\sigma_c^-=\{\omega\in\mathbb{R};\omega=(\alpha-\beta)x,x\in[-L_1,L_2]\}. \tag{24}$$

We denote the spectrum of the evolution equation by $\sigma_c=\sigma_c^+\cup\sigma_c^-$, for there is no other spectrum. This linear system is stable with regard to the dispersion relation (all ω are real numbers).

Because the spectrum σ_c is a bounded set, we can apply the Laplace transform defined by

$$\mathcal{L}[\tilde{v}_x(x,t)]:=\int_0^\infty \tilde{v}_x(x,t)e^{i\Omega t}\,dt, \tag{25}$$

where $\Omega\in\mathbb{C}$ must satisfy $\mathrm{Im}(\Omega)>0$ and σ_c lies on the real axis of the complex Ω plane.

By making use of the master-slave structure of (17) [or (16)], we can solve the master equations independent of the slave equations. In Secs. III A–III E, we will evaluate the asymptotic behavior of $\tilde{v}_x$ and $\tilde{b}_x$ instead of $\mathcal{M}_\pm$, since the master equations of (16) are simpler than that of (17). Using this result, the slave equations will be solved in Sec. III E.

A. Fourier–Laplace analysis of master equations

In terms of $\tilde{V}(x,\Omega)=\mathcal{L}[\tilde{v}_x(x,t)]$ and $\tilde{B}(x,\Omega)=\mathcal{L}[\tilde{b}_x(x,t)]$, the master equations are transformed to

$$(\Omega-\alpha x)\Delta\tilde{V}=-\beta x\Delta\tilde{B}+\Delta\phi,$$

$$(\Omega-\alpha x)\tilde{B}=-\beta x\tilde{V}+\psi, \tag{26}$$

where $\phi(x)=i\tilde{v}_x(x,0)$ and $\psi(x)=i\tilde{b}_x(x,0)$ are holomorphic functions on $[-L_1,L_2]$.

By eliminating $\tilde{B}$, we obtain

$$[(\Omega-\alpha x)^2-\beta^2x^2]\Delta\left(\frac{\tilde{V}}{\Omega-\alpha x}\right)-[2\alpha(\Omega-\alpha x)+2\beta^2 x]\partial_x\left(\frac{\tilde{V}}{\Omega-\alpha x}\right)=\Delta\phi-\beta x\Delta\left(\frac{\psi}{\Omega-\alpha x}\right). \tag{27}$$

This equation becomes singular at $\{(x,\Omega);\Omega-(\alpha\pm\beta)x=0\}$, which is related to the two continuous spectra $\sigma_c^\pm$, but it is important to note that $\{(x,\Omega);\Omega-\alpha x=0\}$ is *not* a singularity of (27) (see Appendix D). In general, we often encounter this apparent singularity concurrent with the elimination of variables.[15]

Let us suppose that we have a solution of (27). By the Cauchy–Kovalevsky theorem, the solution $\tilde{V}(x,\Omega)$ must be a holomorphic function in

$$(x,\Omega)\in([-L_1,L_2]\times\mathbb{C})\setminus\{(x,\Omega);\Omega-(\alpha\pm\beta)x=0\}. \tag{28}$$

Given that $[-L_1,L_2]\times(\mathbb{C}\setminus\sigma_c)$ is a subset of (28), this $\tilde{V}(x,\Omega)$ is holomorphic for all $x\in[-L_1,L_2]$ as far as $\Omega\in\mathbb{C}\setminus\sigma_c$ (resolvent set).

The relation between the Laplace transform and the Fourier transform is made clear in the hyperfunction theory (see Appendix B). Since the spectrum σ_c is bounded, the inverse Laplace transform is equivalent to the Dunford integral (or the double Bromwich integral according to Sedláček[6]):

$$\tilde{v}_x(x,t)=-\frac{1}{2\pi}\oint_{\mathcal{C}(\sigma_c)}\tilde{V}(x,\Omega)e^{-i\Omega t}\,d\Omega, \tag{29}$$

where the integral path $\mathcal{C}(\sigma_c)$ encircles the spectrum σ_c counterclockwise. By deforming $\mathcal{C}(\sigma_c)$ into the vicinity of σ_c, we may write

$$\tilde{v}_x(x,t)=\int_{\sigma_c}\hat{\tilde{v}}_x(x,\omega)e^{-i\omega t}\,d\omega, \tag{30}$$

where

$$\hat{\tilde{v}}_x(x,\omega)=\frac{1}{2\pi}[\tilde{V}(x,\omega+i0)-\tilde{V}(x,\omega-i0)]. \tag{31}$$

This $\hat{\tilde{v}}_x(x,\omega)$ on $[-L_1,L_2]\times\sigma_c$ corresponds to the Fourier transform of $\tilde{v}_x(x,t)$ in a generalized sense, i.e., $\hat{\tilde{v}}_x(x,\omega)$ is a singular eigenfunction defined by the hyperfunction theory.

B. Solutions near singular points

The singularity of $\tilde{V}(x,\Omega)$ can be investigated by the Frobenius method.[16] However, since (27) has complicated inhomogeneous terms, we need more careful treatment to obtain the particular solution and understand its dependence on Ω.

We perform the series expansion of the equation in the neighborhood of two singular points, $x=X^+(\Omega)$ and $x=X^-(\Omega)$, which are defined as $X^\pm(\Omega)=\Omega/(\alpha\pm\beta)$. In the following calculation, we solve both cases simultaneously, using the double signs $x=X^\pm$. Let $\xi=(x-X^\pm)/r$ be a local variable that is scaled by

$$r = X^+ - X^- = \frac{2\beta\Omega}{\beta^2 - \alpha^2}. \tag{32}$$

In terms of this variable, (27) is transformed into

$$\xi^2 \Delta_\xi \widetilde{U} + \xi \frac{2\xi \pm 1}{\xi \pm 1} \partial_\xi \widetilde{U} = \Phi(x,\Omega), \tag{33}$$

where $\Delta_\xi = \partial_\xi^2 - r^2k^2$ and

$$\widetilde{U}(x,\Omega) = r\frac{\widetilde{V}(x,\Omega)}{\Omega - \alpha x} = \frac{\widetilde{V}(x,\Omega)}{\alpha(\gamma_2 - \xi)}, \tag{34}$$

$$\Phi(x,\Omega) = \frac{\xi}{(\alpha^2 - \beta^2)(\xi \pm 1)r} \times \left[\Delta_\xi \phi + \beta(\gamma_1 - \xi)\Delta_\xi \left(\frac{\psi}{\alpha(\gamma_2 - \xi)} \right) \right], \tag{35}$$

$$\gamma_1 = \frac{1}{2}\left(\frac{\alpha}{\beta} \mp 1\right), \quad \gamma_2 = \frac{1}{2}\left(\frac{\beta}{\alpha} \mp 1\right). \tag{36}$$

In contrast to the normal modes analysis such as (18), we have the complicated inhomogeneous terms $\Phi(x,\Omega)$.

First, we solve the homogeneous equation of (33) by setting $\Phi(x,\Omega) \equiv 0$. According to the Frobenius method,[16] the substitution of the series expansion $\widetilde{U}(\xi;\Omega) = \sum_{n=0}^{\infty} \widetilde{U}_n(\Omega)\xi^{n+\lambda}$ [where $\widetilde{U}_0(\Omega) \neq 0$] yields

$$\xi^\lambda \sum_{m=0}^{\infty} \sum_{n=0}^{m} \widetilde{U}_n(\Omega) f_{m-n}(n+\lambda)\xi^m = 0, \tag{37}$$

where

$$f_0(\lambda) = \lambda^2, \quad f_1(\lambda) = \pm\lambda, \quad f_2(\lambda) = -r^2k^2 - \lambda, \tag{38}$$

and

$$f_j(\lambda) = -(\mp 1)^j \lambda \quad \text{for } j \geqslant 3. \tag{39}$$

We can determine $\{\widetilde{U}_n(\Omega); n=1,2,\ldots\}$ recursively by equating the coefficients of ξ^m in (37) with zero. Since the indicial equation $f_0(\lambda)=0$ has a repeated root $\lambda=0$, we obtain a regular solution

$$\widetilde{U}_r(\xi;\Omega) = 1 + \frac{r^2k^2}{4}\xi^2 \mp \frac{r^2k^2}{18}\xi^3 + \cdots, \tag{40}$$

and a singular solution

$$\widetilde{U}_s(\xi;\Omega) = \widetilde{U}_r(\xi;\Omega)\mathrm{Log}\ \xi + \left[\mp\xi - \frac{1}{2}\left(\frac{r^2k^2}{2} - 1\right) \xi^2 \mp \frac{1}{3}\left(\frac{5r^2k^2}{36} + 1\right)\xi^3 + \cdots \right], \tag{41}$$

where we set $\widetilde{U}_0(\Omega) \equiv 1$. The radius of convergence is found to be $|r| = |X^+ - X^-|$, which corresponds to the distance between the two singular points. Supposing that there is a complex plane X with $\mathrm{Re}(X) = x$ as shown in Fig. 1, we denote the interior of the circle of convergence simply by

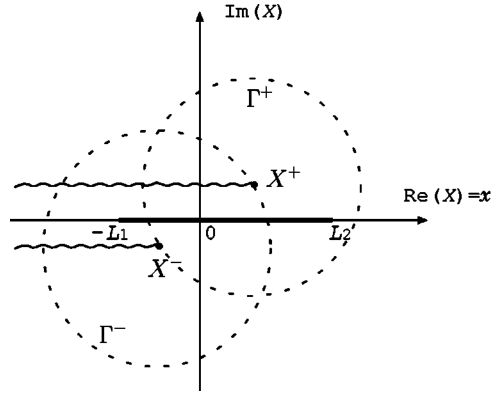

FIG. 1. Singular points $X=X^+$ and $X=X^-$ in the complex X plane. The Frobenius method develops the solutions in series near these points. The circles of convergence are indicated by dotted lines, and the inner regions are, respectively, denoted by Γ^+ and Γ^-. Wavy lines represent the branch cuts of the logarithmic singularities.

$$\Gamma^\pm(\Omega) = \{X \in \mathbb{C}; |X - X^\pm| < |r|\}. \tag{42}$$

The logarithmic function of complex variable, Log X, is defined on a Riemann surface $\{X \in \mathbb{C}; -\pi < \arg X < \pi\}$, which has a discontinuity on a real axis; we will use the formula

$$\mathrm{Log}(x \pm i0) = \log|x| \pm \pi i Y(-x) \quad (x \in \mathbb{R}), \tag{43}$$

where $Y(x)$ denotes the Heaviside function.[3]

Next, we take into account the inhomogeneous term $\Phi(x,\Omega)$, which includes the arbitrary initial conditions ϕ and ψ. The particular solution (denoted by $\widetilde{U}_p$) will be solved for all $\Omega \in \mathbb{C}\backslash\{0\}$, based on the fact that $\Phi(x,\Omega)$ diverges at $\Omega = 0$ (or $r=0$). Substituting arbitrary holomorphic functions $\phi(x) = \sum_{n=0}^{\infty}\phi_n r^n \xi^n$ and $\psi(x) = \sum_{n=0}^{\infty}\psi_n r^n \xi^n$, we can also expand the inhomogeneous term as $\Phi = \sum_{n=1}^{\infty}\Phi_n(\Omega)\xi^n$, where, for instance,

$$\Phi_1(\Omega) = \pm \frac{1}{\alpha^2 - \beta^2}\left[(2r\phi_2 - rk^2\phi_0) \mp \left(2r\psi_2 - rk^2\psi_0 + \frac{2\psi_1}{\gamma_2} + \frac{2\psi_0}{r\gamma_2^2}\right)\right]. \tag{44}$$

Let us substitute a holomorphic function $\widetilde{U}_p(\xi;\Omega) = \sum_{n=0}^{\infty}\widetilde{U}_n(\Omega)\xi^n$ into (33). By comparing the coefficients of $\xi^n (n=0,1,\ldots)$, we obtain

$$\begin{pmatrix} 0 & & & & \\ f_1(0) & f_0(1) & & & \\ f_2(0) & f_1(1) & f_0(2) & & \\ f_3(0) & f_2(1) & f_1(2) & f_0(3) & \\ \vdots & & & & \ddots \end{pmatrix} \begin{pmatrix} \widetilde{U}_0 \\ \widetilde{U}_1 \\ \widetilde{U}_2 \\ \widetilde{U}_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \vdots \end{pmatrix}. \tag{45}$$

In these relations, we may fix $\widetilde{U}_0(\Omega) \equiv 0$ to remove the indefiniteness of the homogeneous solution $\widetilde{U}_r(\xi;\Omega)$ from the particular solution. Then we can determine $\{\widetilde{U}_n(\Omega); n = 1,2,\ldots\}$ uniquely for any $\{\Phi_n(\Omega); n=1,2,\ldots\}$, due to the fact that the diagonal components $f_0(1), f_0(2), \ldots$ are not zero and are different from each other. In other words, the particular solution $\tilde{U}_p(\xi;\Omega)$ is found to be holomorphic at $\xi=0$ for any initial conditions. Although this $\tilde{U}_p$ might be singular at $\xi=\gamma_2$, we already know that $\xi=\gamma_2$ is the apparent singularity, and therefore $\tilde{V}_p=\alpha(\gamma_2-\xi)\tilde{U}_p$ must be holomorphic in $\Gamma^{\pm}(\Omega)$.

C. Singularity at $\omega\in\sigma_c\backslash\{0\}$

Using $\tilde{V}_{r,s,p}(\xi;\Omega)=\alpha(\gamma_2-\xi)\tilde{U}_{r,s,p}(\xi;\Omega)$, the general solution is represented by

$$\tilde{V}(x,\Omega)=C_r^{\pm}(\Omega)\tilde{V}_r(\xi;\Omega)+C_s^{\pm}(\Omega)\tilde{V}_s(\xi;\Omega) + \tilde{V}_p(\xi;\Omega) \quad \text{in } \Gamma^{\pm}(\Omega). \tag{46}$$

The coefficients $C_r^{\pm}(\Omega)$ and $C_s^{\pm}(\Omega)$ are determined by the following consideration. For $\Omega\in\mathbb{C}\backslash\sigma_c$, the domain of $\tilde{V}(x,\Omega)$ can be extended to (28) by the analytic continuation. By imposing the boundary condition

$$\tilde{V}(-L_1,\Omega)=0, \quad \tilde{V}(L_2,\Omega)=0, \tag{47}$$

the coefficients $C_r^{\pm}(\Omega)$ and $C_s^{\pm}(\Omega)$ are uniquely determined, for Ω belongs to the resolvent set $\mathbb{C}\backslash\sigma_c$.

Let us investigate the limits of $\tilde{V}(x,\omega+i0)$ and $\tilde{V}(x,\omega-i0)$ for $\omega\in\sigma_c\backslash\{0\}$ (the case $\omega=0$ will be discussed later). In this case, the two singular points X^+ and X^- approach, respectively, $x^+:=\omega/(\alpha+\beta)$ and $x^-:=\omega/(\alpha-\beta)$. Since $\tilde{V}_s$ has a branch cut in the complex X plane as shown in Fig. 1, there is discontinuity due to the logarithmic term [see (43)],

$$\tilde{V}_s(\xi;\omega+i0)\neq\tilde{V}_s(\xi;\omega-i0), \tag{48}$$

which causes, in general,

$$C_r^{\pm}(\omega+i0)\neq C_r^{\pm}(\omega-i0), \quad C_s^{\pm}(\omega+i0)\neq C_s^{\pm}(\omega-i0). \tag{49}$$

Then the Fourier transform $\hat{\tilde{v}}_x(x,\omega)$ given by (31) has singularities of $\log|x-x^{\pm}|$ and $Y(x-x^{\pm})$. This result is qualitatively the same as the conventional Alfvén singularity without flow which was studied by Barston,[5] Hasegawa and Uberoi.[12] The inverse Laplace transform leads to phase-mixing damping $\propto 1/t$ (see also Ref. 7).

Using this result, $\tilde{B}(x,\Omega)$ is easily obtained by

$$\tilde{B}(x,\Omega)=\frac{-\beta x\tilde{V}(x,\Omega)+\psi(x)}{\Omega-\alpha x}. \tag{50}$$

For the limit of $\Omega\to\omega\in\sigma_c\backslash\{0\}$, the singularity of $\tilde{B}(x,\Omega)$ is essentially the same as $\tilde{V}(x,\Omega)$.

D. Singularity at $\omega=0$

Within the continuous spectra, the case of $\omega=0$ requires more careful analysis. When $\Omega\to 0$, the two singular points $x=X^+$ and $x=X^-$ collide with each other on the rational surface $x=0$, and the radius of convergence $|r|$ becomes zero. In this case, we must deal with the two singularities at once to obtain the solution $\tilde{V}(x,\Omega)$ which is consistent in both regions $\Gamma^+(\Omega)$ and $\Gamma^-(\Omega)$. For this purpose, we start our analysis with the solution near X^+ and, from this point of view, observe the other singularity at X^-. Noting the formula

$$\mathrm{Log}(1+X)=X-\frac{X^2}{2}+\frac{X^3}{3}-\cdots \quad (X\in\mathbb{C}), \tag{51}$$

we can rewrite the homogeneous solutions in $\Gamma^+(\Omega)$ as

$$\tilde{U}_r(\xi;\Omega)=1+r^2\left[\frac{k^2}{4}\xi^2-\frac{k^2}{18}\xi^3+\ldots\right] \tag{52}$$

$$=1+O(\Omega), \tag{53}$$

$$\tilde{U}_s(\xi;\Omega)=\tilde{U}_r(\xi;\Omega)\mathrm{Log}\;\xi-\mathrm{Log}\;(1+\xi) + r^2\left[-\frac{k^2}{4}\xi^2-\frac{5k^2}{108}\xi^3-\ldots\right] \tag{54}$$

$$=\mathrm{Log}\frac{x-X^+}{x-X^-}+O(\Omega), \tag{55}$$

where $\xi=(x-X^+)/r$, and $O(\Omega)$ represents the terms that converge to zero uniformly in $\Gamma^+(\Omega)$ for the limit of $\Omega\to 0$ (or $r\to 0$). The function (55) describes how two logarithmic singularities collide in this limit.

At the same time, for $\Omega\to 0$, the inhomogeneous term $\Phi(x,\Omega)$ diverges in proportion to $1/\Omega$ unless $\psi_0=0$. By multiplying both sides of (33) by Ω, we obtain

$$\xi^2\Delta_\xi(\Omega\tilde{U})+\xi\frac{2\xi\pm 1}{\xi\pm 1}\partial_\xi(\Omega\tilde{U}) = -\frac{\xi}{\xi\pm 1}\frac{\gamma_1-\xi}{\alpha(\gamma_2-\xi)^3}\psi_0+O(r). \tag{56}$$

After some considerations, we find that

$$\Omega\tilde{U}_p(\xi;\Omega)=\frac{\psi_0}{\beta(\gamma_2-\xi)}+O(r) \tag{57}$$

is a particular solution. It follows that

$$\Omega\tilde{V}_p(\xi;\Omega)=\frac{\alpha}{\beta}\psi_0+O(r) \text{ in } \Gamma^{\pm}(\Omega). \tag{58}$$

From these results, we successfully extract the singularity of $\tilde{V}(x,\Omega)$ in the neighborhood of $\Omega=0$. The general solution near $x=X^+$ is represented by

$$\Omega\tilde{V}(x,\Omega)=C_r^+(\Omega)(\Omega-\alpha x)+C_s^+(\Omega)(\Omega-\alpha x)\mathrm{Log}\frac{x-X^+}{x-X^-} + \frac{\alpha}{\beta}\psi_0+O(\Omega) \text{ in } \Gamma^+(\Omega). \tag{59}$$

Substituting this into (26), we get

$$\Omega\tilde{B}(x,\Omega)=-C_r^+(\Omega)\beta x-C_s^+(\Omega)\beta x\;\mathrm{Log}\frac{x-X^+}{x-X^-}+\psi_0 + O(\Omega) \text{ in } \Gamma^+(\Omega). \tag{60}$$

These Laplace-transformed variables $\tilde{V}(x,\Omega)$ and $\tilde{B}(x,\Omega)$ have the same singularity and include $1/\Omega$ which must be

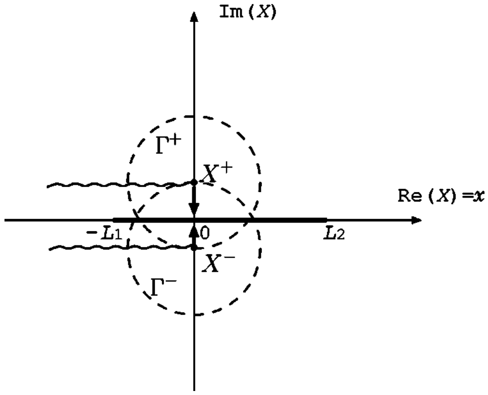

FIG. 2. In the case of $\beta > |\alpha| > 0$, X^{+} and X^{-} approach the x axis from, respectively, the upper and lower half-planes when $\Omega \to +i0$.

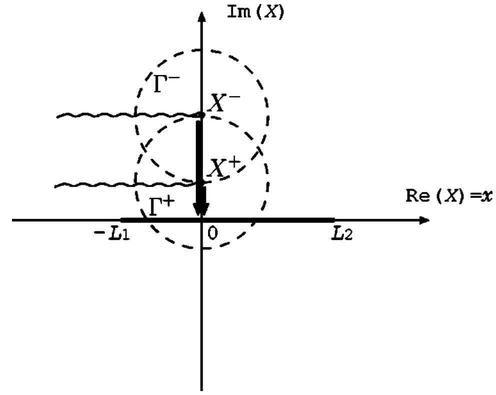

FIG. 3. In the case of $\alpha > \beta > 0$, X^{+} and X^{-} approach the x axis from the same direction when $\Omega \to +i0$.

distinguished from an isolated pole in that it has the logarithmic singularities in the numerator. This complicated singularity at $\Omega=0$ is closely related to the magnitude relation of α and β.

If the magnetic shear is stronger than the flow shear ($|\alpha|<|\beta|$), the two singular points X^{+} and X^{-} approach the origin from opposite sides of the $x=\mathrm{Re}(X)$ axis for the limit of $\Omega \to \pm i0$ (see Fig. 2). Let us assume $\beta>0$ (the case of $\beta<0$ can be discussed in the same manner). The definition $X^{\pm}(\Omega)=\Omega/(\alpha\pm\beta)$ implies that the limit of $\Omega\to\pm i0$ corresponds to $X^{+}\to\pm i0$ and $X^{-}\to\mp i0$. Then we get

$$[\Omega\tilde{V}(x,\Omega)]_{\Omega\to\pm i0}$$

$$= - C_r^{+}(\pm i0)\alpha x - C_s^{+}(\pm i0)\alpha x \mathrm{Log}\frac{x \mp i0}{x \pm i0} + \frac{\alpha}{\beta}\psi_0 \tag{61}$$

$$=- C_r^{+}(\pm i0)\alpha x \pm C_s^{+}(\pm i0)\alpha x 2\pi i Y(-x) + \frac{\alpha}{\beta}\psi_0. \tag{62}$$

It is important to note that this equality is valid only in the infinitely small region $\Gamma^{+}(\pm i0)$. The whole solution in $[-L_1,L_2]$ will be attained by the analytic continuation which is always possible in the domain (28). By multiplying the equations in (26) by Ω and taking the limit of $\Omega\to 0$, we note that

$$\Delta[\Omega\tilde{V}(x,\Omega)]_{\Omega\to\pm i0} = 0 \quad \text{in } [-L_1,L_2]\setminus\{0\} \tag{63}$$

must be satisfied except in the case of $x=0$. The solutions of (63), a linear combination of e^{kx} and e^{-kx}, can be connected with the local solution (62) at $x=0$, which determines $C_r^{+}(\pm i0)$ and $C_s^{+}(\pm i0)$. Taking the boundary condition into account, we obtain

$$[\Omega\tilde{V}(x,\Omega)]_{\Omega\to\pm i0} = \frac{\alpha}{\beta}\psi_0 g(x), \tag{64}$$

where

$$g(x) := \begin{cases} \dfrac{\sinh(-kL_1 - kx)}{\sinh(-kL_1)} & (-L_1 \leqslant x \leqslant 0), \\ \dfrac{\sinh(kL_2 - kx)}{\sinh(kL_2)} & (0 < x \leqslant L_2). \end{cases} \tag{65}$$

Similarly, we obtain

$$[\Omega\tilde{B}(x,\Omega)]_{\Omega\to\pm i0} = \psi_0 g(x). \tag{66}$$

By the inverse Laplace transform, the asymptotic behavior ($t\to\infty$) is represented by

$$\tilde{v}_x(x,t) \to \frac{\alpha}{\beta}\tilde{b}_x(0,0)g(x), \tag{67}$$

$$\tilde{b}_x(x,t) \to \tilde{b}_x(0,0)g(x), \tag{68}$$

where we used $\psi_0 = i\tilde{b}_x(0,0)$. The convergence speed is characterized by $\propto 1/t$ according to the result in the preceding section. We conclude that $\tilde{v}_x(x,t)$ and $\tilde{b}_x(x,t)$ converge to functions that have a derivative jump at $x=0$. Especially, the value of $\tilde{b}_x(x,t)$ at $x=0$ remains constant, which can be confirmed by viewing (26) only at $x=0$. Even if we consider the case of $\alpha=0$ (no flow shear), the above result is applicable and we obtain $\tilde{v}_x(x,t)\to 0$ instead of (67).

When the flow shear exceeds the magnetic shear ($|\alpha| > |\beta|$), the singularities of (59) and (60) change drastically. Since the signs of $\mathrm{Im}(X^{+})$ and $\mathrm{Im}(X^{-})$ are the same, the two singular points approach the origin from the same side of the x axis. For example, in the case of $\alpha>\beta>0$, both X^{+} and X^{-} approach from the upper half-plane for $\Omega\to i0$, and from the lower half-plane for $\Omega\to -i0$ (see Fig. 3). Then one of the two linearly independent solutions in (59) disappears due to

$$\mathrm{Log}\frac{x - X^{+}}{x - X^{-}} \to \mathrm{Log}\frac{x \mp i0}{x \mp i0} \equiv 0 \quad (\Omega \to \pm i0). \tag{69}$$

This situation must be avoided by the divergence of the coefficient $C_s^{+}(\Omega)\propto 1/\Omega$, which recovers a linearly independent solution. It follows that

$$[\Omega\tilde{V}(x,\Omega)]_{\Omega\to\pm i0} = -C_r^+(\pm i0)\alpha x - C_1(\pm i0)\alpha[\chi(x)-1] + \frac{\alpha}{\beta}\psi_0 \quad \text{in } \Gamma^+(\pm i0), \tag{70}$$

where we set $C_1(\Omega)=rC_s^+(\Omega)$ and

$$\chi(x) := \begin{cases} 1 & (x=0), \\ 0 & (x\neq 0). \end{cases} \tag{71}$$

The continuation between the solution of (63) and (70) results in

$$[\Omega\tilde{V}(x,\Omega)]_{\Omega\to\pm i0} = \frac{\alpha}{\beta}\psi_0\chi(x). \tag{72}$$

Therefore, the inverse Laplace transform gives, for $t\to\infty$,

$$\tilde{v}_x(x,t) \to \frac{\alpha}{\beta}\tilde{b}_x(0,0)\chi(x), \tag{73}$$

$$\tilde{b}_x(x,t) \to \tilde{b}_x(0,0)\chi(x). \tag{74}$$

We conclude that $\tilde{v}_x(x,t)$ and $\tilde{b}_x(x,t)$ converge to zero almost everywhere.

E. Algebraic growths of slave variables

Finally, the slave equations are solved by using the result of $\tilde{V}(x,\Omega)$. The Laplace transform of $\mathcal{M}_\pm=\tilde{v}_x\mp\tilde{b}_x$ is given by

$$\mathcal{L}[\mathcal{M}_\pm](x,\Omega) = \frac{-(\alpha\mp\beta)(x-X^\mp)\tilde{V}(x,\Omega)\mp\psi(x)}{\Omega-\alpha x}. \tag{75}$$

We note that either of the two logarithmic singularities, $\mathrm{Log}(x-X^+)$ or $\mathrm{Log}(x-X^-)$, of $\tilde{V}(x,\Omega)$ is smoothed by the multiplication of $(x-X^\mp)$. Plugging this into the Laplace-transformed slave equations, we obtain

$$\mathcal{L}[\mathcal{S}_\pm] = -\frac{\iota_\mp(x)}{\Omega-\alpha x}\left[\tilde{V}(x,\Omega)\pm\frac{\psi(x)}{\Omega-(\alpha\pm\beta)x}\right] + i\frac{\mathcal{S}_\pm(x,0)}{\Omega-(\alpha\pm\beta)x}. \tag{76}$$

The inverse Laplace transform of the last term on the right-hand side yields the solution

$$\mathcal{S}_\pm(x,0)e^{-i(\alpha\pm\beta)xt}, \tag{77}$$

which shows the phase mixing due to the Alfvén continuum and always exists even if the master variables are initially zero.

The first term in (76) represents the response to the master variables. The singularity of this term at $\Omega=\omega\in\sigma_c\backslash\{0\}$ does not yield algebraic growth of $\mathcal{S}_\pm$. However, there is a strong singularity at $\Omega=0$, for the substitution of (59) leads to

$$\mathcal{L}[\mathcal{S}_\pm] = -\frac{\iota_\mp(x)}{\Omega}\left[C_r^+(\Omega)+C_s^+(\Omega)\mathrm{Log}\frac{x-X^+}{x-X^-}+\frac{\psi_0}{\beta(X^\pm-x)} + O(\Omega)\right] + i\frac{\mathcal{S}_\pm(x,0)}{\Omega-(\alpha\pm\beta)x} \quad \text{in } \Gamma^+(\Omega). \tag{78}$$

For $\iota_\mp(0)\neq 0$ and $\psi_0=i\tilde{b}_x(0,0)\neq 0$, the singularity of $1/\Omega(X^\pm-x)$ brings about a localized algebraic growth at $x=0$. The most dominant asymptotic behavior is, therefore, estimated by

$$\mathcal{S}_\pm(x,t) \sim \frac{1-e^{-i(\alpha\pm\beta)xt}}{\beta x}\iota_\mp(0)\tilde{b}_x(0,0). \tag{79}$$

By going back to $\tilde{w}_x=(\mathcal{S}_++\mathcal{S}_-)/2$ and $\tilde{j}_x=(\mathcal{S}_--\mathcal{S}_+)/2$, we can conclude that $\tilde{w}_x$ and $\tilde{j}_x$ increase remarkably near $x=0$. Given that we supposed $\mathbf{k}\cdot\mathbf{B}(0)=0$, the parallel components, $\tilde{v}_\parallel$ and $\tilde{b}_\parallel$, with respect to $\mathbf{B}$ increase in proportion to t, as follows:

$$\tilde{v}_\parallel(0,t) = \tilde{w}_x(0,t)/ik \propto t, \tag{80}$$

$$\tilde{b}_\parallel(0,t) = \tilde{j}_x(0,t)/ik \propto t. \tag{81}$$

If there is no shear flow ($\alpha=0$), the growths of $\mathrm{Re}(\tilde{w}_x)$ and $\mathrm{Im}(\tilde{j}_x)$ vanish and only the parallel motion $\tilde{v}_\parallel$ increases in proportion to t.

By viewing the generator of (17), we note that, among the four Alfvén waves, $\mathcal{M}_+$ (or $\mathcal{M}_-$) acts on $\mathcal{S}_-$ (or $\mathcal{S}_+$) that propagates in the opposite direction. That is why the resonant interaction occurs only at the zero frequency $\omega=0$, while the four continuous spectra degenerate in the region $\sigma_c^+\cap\sigma_c^-$.

IV. SUMMARY

The system of linearized equations governing the fluctuations in ideal MHD has a non-self-adjoint generator with essential singularities. The resonant interaction among the four Alfvèn continuous spectra causes an algebraic instability localized on the rational surface $x=x_r$ where $\mathbf{k}\cdot\mathbf{B}(x_r)=0$. The initial fluctuation of magnetic field on the rational surface, i.e., $\tilde{b}_x(x_r,0)$, plays an essential role, which has been assumed to be zero in the stability analysis of the Lagrange displacement [see (4)].

The physical mechanism of this instability can be understood as follows. If we have an ambient magnetic shear and no flow, the fluctuation on the rational surface does not inflect the magnetic field line and remains temporally constant because no force acts on it. In the equation of motion, the electromagnetic force produced by this fluctuation $\tilde{b}_x(x_r,0)$ and the ambient current constantly accelerates the parallel motion $\tilde{v}_\parallel(x_r,t)$ with respect to $\mathbf{B}$, or, equivalently, the ambient pressure gradient along the magnetic field accelerates it, which yields the algebraic growth, $\tilde{v}_\parallel(x_r,t)\propto t$. In order to avoid this algebraic instability, it is required that the condition $[\mathbf{B}'(x_r)\times\mathbf{k}]\cdot\mathbf{e}_x=0$ must be satisfied for arbitrary wave number $\mathbf{k}$, which is equivalent to the condition

$$\left(\frac{|\mathbf{B}|^2}{2}\right)'(x_r) = -P'(x_r) = 0. \tag{82}$$

Therefore, the pressure gradient on the rational surface must be zero.

In the presence of flow, this algebraic behavior undergoes a Doppler shift, $\tilde{v}_\parallel(x_r,t)\propto te^{-i\omega_r t}$, where $\omega_r=\mathbf{k}\cdot\mathbf{V}(x_r)$. The advection of fluctuations by mean shear flow yields the

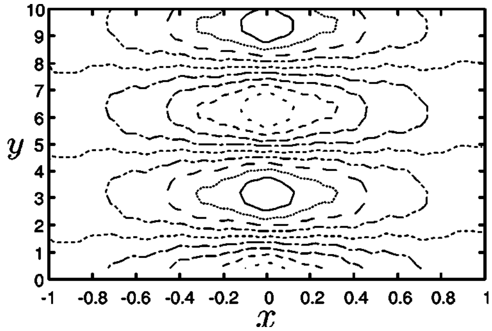

FIG. 4. Contour plot of $\tilde{b}_x(x,t)e^{ik_y y}$ at $t=40$ with the parameters $\alpha=0.5$, $\beta=1$, $k_y=1$, and $[-L_1,L_2]=[-1,1]$. This is a typical numerical solution of the initial value problem (16).

algebraic growth of $\tilde{b}_\parallel(x_r,t)\propto t$ in addition to $\tilde{v}_\parallel(x_r,t)\propto t$. The algebraic instability we have shown can be more important than that of Lau and Liu,[17,18] for the growth of the latter instability is transient and saturated in a finite time.

Asymptotic behavior of the master variables, $\tilde{v}_x(x,t)$ and $\tilde{b}_x(x,t)$, changes discontinuously when the flow shear (α) exceeds the magnetic shear (β). It is interesting to note that for large B_z(=const) and small k_z, the master variables are, respectively, similar to the stream function and the flux function in the reduced MHD theory (see Appendix C). For weak flow shear $(|\alpha|<|\beta|)$, the asymptotic behavior given by (67) and (68) represents the generation of a magnetic island as shown in Fig. 4. On the other hand, for strong flow shear $(|\alpha|>|\beta|)$, the asymptotic behavior makes the transition to (73) and (74), where the stretching effect of shear flow destroys the magnetic island as shown in Fig. 5. The ratio β^2/α^2 is called the magnetic Richardson number, which is known as an important parameter also in the normal modes analysis.[11] In real plasmas, such a time evolution into a singular structure is avoided by the dissipation effect. Our work is, however, useful for understanding the universal behavior of fluctuation; the generation of a magnetic island and the stretching effect of shear flow. We clarified that the structure of the magnetic island is closely related to the Alfvén continuous spectrum and is identified as the coalescence of two Alfvén singularities.

From a mathematical point of view, if the initial value problem of a non-self-adjoint system is solved by the Laplace transform, complicated singularities far from simple poles appear in the Dunford integral path. The translation from the initial value problem into the normal modes analysis is generally difficult because it requires the spectral resolution of a non-self-adjoint operator, and we must generalize the notion of *nilpotent* for degenerate continuous spectra. In our model, the algebraic behavior of the solution arose from the coalescence of Alfvén singularities. The rigorous spectral theory for this non-self-adjoint system will be discussed elsewhere.

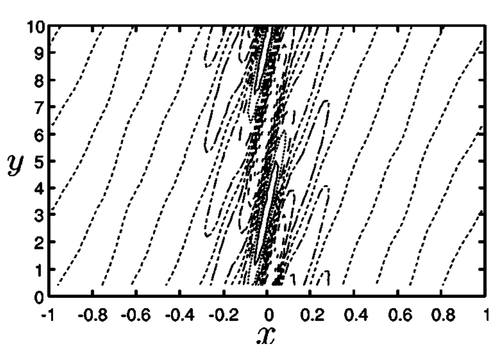

FIG. 5. Contour plot of $\tilde{b}_x(x,t)e^{ik_y y}$ at $t=40$ with the parameters $\alpha=1$, $\beta=0.5$, $k_y=1$, and $[-L_1,L_2]=[-1,1]$.

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APPENDIX A: NON-SELF-ADJOINTNESS OF EVOLUTION EQUATION

In general, let us write a linear system in the form of an evolution equation: $i\partial_t f=\mathcal{K}f$ where f denotes a variable and $\mathcal{K}$ a linear operator.

When the space of variables has a finite dimension n, and $\mathcal{K}$ is a *matrix*, the solution is definitely given by the linear algebra as follows. A matrix is called *semisimple* if it can be transformed into a diagonal matrix,

$$P^{-1}\mathcal{K}P=\begin{pmatrix}\omega_1 & & & \\ & \omega_2 & & \\ & & \ddots & \\ & & & \omega_n\end{pmatrix}, \tag{A1}$$

by using a set of eigenvectors $P=(\mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_n)$, where $\omega_1,\omega_2,\ldots,\omega_n$ may include repeated eigenvalues. A self-adjoint matrix is always semisimple. For a semisimple generator $\mathcal{K}$, the solution is represented by a linear combination of the exponential behavior $\mathbf{v}_j e^{-i\omega_j t}$.

A general matrix does not assume the diagonalized form (A1), while it can be always cast into a Jordan canonical form,[19]

$$P^{-1}\mathcal{K}P=\begin{pmatrix}\omega_1 I+N & & & \\ & \omega_2 I+N & & \\ & & \ddots & \\ & & & \omega_m I+N\end{pmatrix}, \tag{A2}$$

where generally $m\leqslant n$. The $\omega_j I+N$ appearing on the diagonal line denotes a $n_j\times n_j$, block,

$$\omega_j I+N=\begin{pmatrix}\omega_j & 1 & & \\ & \ddots & \ddots & \\ & & \omega_j & 1\\ & & & \omega_j\end{pmatrix}, \tag{A3}$$

where I denotes unit matrix and N is called the *nilpotent* matrix. In other words, a general matrix can be decomposed into a semisimple part and a nilpotent part, and the diagonalization fails due to the latter. In the presence of the $n_j\times n_j$

block, corresponding eigenvectors show algebraic behavior such as $t^k e^{-i\omega_j t}$ ($k=0,1,2,\ldots,n_j-1$). Physically, the algebraic growth of amplitudes implies the *resonance* of oscillators with a common frequency. A nilpotent matrix is, thus, the mathematical representation of the resonant interactions in degenerate (common eigenvalue) modes.

In the case of infinite dimensional (functional) space, the Von Neumann's theorem[10] enables spectral resolution of any self-adjoint operator $\mathcal{K}$, where we may observe a continuous spectrum in addition to point spectra. However, we do not have a general spectral theory for non-self-adjoint operators in functional space. The resonant interaction among degenerate continuous spectra yields much more spatially and temporally complicated behavior than degenerate point spectra.[2]

In plasma physics, inhomogeneity of equilibrium causes a continuous spectrum (e.g., shear flow,[20] magnetic shear,[7] gradient of plasma density,[5,6] and so on). Then, in the normal modes analysis, the eigenvalue problem becomes a differential equation and the continuous spectrum is related to a singular point. We can formally obtain singular eigenfunctions corresponding to the continuous spectrum (for example, Van Kampen mode[21]). Although Case[22] and Sedláček[6] discussed the equivalence of two approaches, the normal modes analysis and the initial value problem, the continuous spectrum they considered seems to be *semisimple*. In this paper, we solve an initial value problem by using the Laplace transform; the normal modes analysis is not well defined for degenerate continuous spectra that might conceal a *nilpotent* part.

APPENDIX B: HYPERFUNCTION THEORY

Generalized functions (such as delta functions) are defined by two approaches; Schwartz's distribution theory and Sato's hyperfunction theory. We invoke the latter in this paper because it clarifies the relation between the Laplace transform and the Fourier transform,[3] which enables us to analyze the continuous spectrum rigorously.

Let $\mathcal{B}(D)$ be the set of all hyperfunctions on the domain $D\subset\mathbb{R}$. We introduce a region $U\subset\mathbb{C}$ in the complex plane such that $D\subset U$ and denote the set of all holomorphic function on U by $\mathcal{O}(U)$. Each hyperfunction in $\mathcal{B}(D)$ is uniquely defined by a element of $\mathcal{O}(U\backslash D)/\mathcal{O}(U)$, i.e., $\mathcal{B}(D) \simeq \mathcal{O}(U\backslash D)/\mathcal{O}(U)$. Formally, we may define a hyperfunction $f(x)\in\mathcal{B}(D)$ by taking the difference of the boundary values of $F(z)\in\mathcal{O}(U\backslash D)$ like

$$f(x) := \lim_{\epsilon\to+0}[F(x+i\epsilon)-F(x-i\epsilon)] \tag{B1}$$

$$= F(x+i0)-F(x-i0), \tag{B2}$$

where $F(z)$ is holomorphic everywhere in U except on D and called *defining function*. For example, the delta function $\delta(x)$ and the Heaviside function $Y(x)$ are defined by $F(z) = -1/(2\pi i z)$ and $F(z)=-\mathrm{Log}(-z)/(2\pi i)$, respectively. The derivative and integral of $f(x)$ are calculated in terms of the defining function,

$$f'(x) := F'(x+i0)-F'(x-i0), \tag{B3}$$

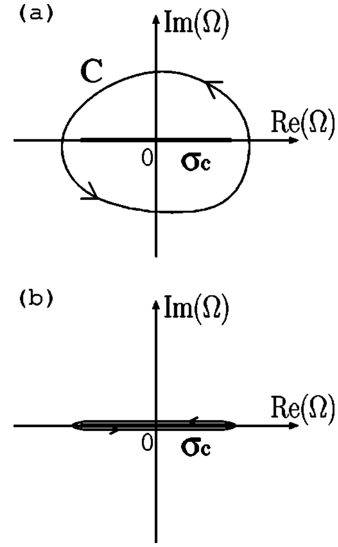

FIG. 6. Translation from (a) the Laplace transform (the initial value problem) into (b) the Fourier transform (the spectral resolution).

$$\int_D f(x)dx := -\oint_{\mathcal{C}(D)} F(z)dz, \tag{B4}$$

where the integral path $\mathcal{C}(D)$ encircles D counterclockwise.

Now we consider a general evolution equation $i\partial_t f=\mathcal{K}f$ as in Appendix A. For simplicity, let $\mathcal{K}$ has a bounded continuous spectrum σ_c, on the real axis. The solution is represented by the Dunford integral (or the inverse Laplace transform) which encircles σ_c [see Fig. 6(a)],

$$f(t) = -\frac{1}{2\pi}\int_{\mathcal{C}(\sigma_c)} i\mathcal{R}(\Omega)f(0)e^{-i\Omega t}\,d\Omega, \quad \Omega\in\mathbb{C}, \tag{B5}$$

$$= -\frac{1}{2\pi}\oint_{\mathcal{C}(\sigma_c)} \hat{F}(\Omega)e^{-i\Omega t}\,d\Omega, \tag{B6}$$

where $\mathcal{R}(\Omega)=(\Omega-\mathcal{K})^{-1}$ denotes the resolvent operator and $\hat{F}(\Omega)=i\mathcal{R}(\Omega)f(0)$ the Laplace transform of $f(t)$. By regarding $\hat{F}(\Omega)/2\pi$ as a defining function, we obtain a hyperfunction $\hat{f}(\omega)\in\mathcal{B}(\sigma_c)$ which satisfies

$$f(t) = \int_{\sigma_c} \hat{f}(\omega)e^{-i\omega t}\,d\omega. \tag{B7}$$

In other words, we deformed $\mathcal{C}(\sigma_c)$ into the vicinity of σ_c [Fig. 6(b)]. The integrand $\hat{f}(\omega)$ is the Fourier transform of $f(t)$ and is expected to represent singular eigenfunction corresponding to the continuous spectrum. Since the spectral theory is not established if $\mathcal{K}$ is non-self-adjoint, we must derive $\hat{f}(\omega)$ rigorously as stated above and the eigenvalue problem $(\omega-\mathcal{K})\hat{f}(\omega)=0$ is not valid.

APPENDIX C: NORMAL VELOCITY AND NORMAL VORTICITY

The normal velocity $\tilde{v}_x$ and the normal vorticity $\tilde{w}_x$ are commonly used in fluid mechanics to represent three dimensional incompressible fluctuations with only one independent variable x.[13] The definition of $\tilde{w}_x = i(k_y\tilde{v}_z - k_z\tilde{v}_y)$ and the condition $\nabla\cdot\tilde{\mathbf{v}}=0$ allow us to calculate the other velocity components,

$$\tilde{v}_y = i\frac{k_y\partial_x\tilde{v}_x + k_z\tilde{w}_x}{k^2}, \quad \tilde{v}_z = i\frac{k_z\partial_x\tilde{v}_x - k_y\tilde{w}_x}{k^2}. \tag{C1}$$

These variables $\tilde{v}_x$ and $\tilde{w}_x$ recover the Clebsch representation in the following special cases. If $k_z=0$, we obtain

$$\tilde{\mathbf{v}} = \frac{\nabla\tilde{v}_x \times \mathbf{e}_z + \tilde{w}_x\mathbf{e}_z}{ik_y}, \tag{C2}$$

and if $k_y=0$,

$$\tilde{\mathbf{v}} = -\frac{\nabla\tilde{v}_x \times \mathbf{e}_y + \tilde{w}_x\mathbf{e}_y}{ik_z}, \tag{C3}$$

where $\tilde{v}_x$ parallels the stream function.

Since the magnetic field is also incompressible, the same representation is applicable to $\tilde{\mathbf{b}}$.

APPENDIX D: APPARENT SINGULARITY

If we eliminate $\tilde{V}$ from (26), we obtain

$$[(\Omega-\alpha x)^2 - \beta^2x^2]\Delta\left(\frac{\tilde{B}}{\beta x}\right) - [2\alpha(\Omega-\alpha x) + 2\beta^2 x]\partial_x\left(\frac{\tilde{B}}{\beta x}\right) = -\Delta\phi + (\Omega-\alpha x)\Delta\left(\frac{\psi}{\beta x}\right) \tag{D1}$$

instead of (27). In this equation, the point $\{(x,\Omega);\Omega-\alpha x=0\}$ is not singular, and as a consequence the solution $\tilde{B}$ must be holomorphic there. By substituting this $\tilde{B}$ into (26), we obtain $\tilde{V}$, which must also be holomorphic at $\{(x,\Omega);\Omega-\alpha x=0\}$. It follows that the condition $\Omega-\alpha x=0$ in (27) does not yield singularity, which is so called the apparent singularity.[16] We can confirm this fact by applying the Frobenius method to (27) near $x=\Omega/\alpha$.

In the same manner, we can find that the point $x=0$ is also an apparent singularity of (D1).

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