Kelvin–Helmholtz instability in Beltrami fields

A. Ito, Z. Yoshida, T. Tatsuno, and S. Ohsaki

Graduate School of Frontier Sciences, The University of Tokyo, Tokyo 113-0033, Japan

S. M. Mahajan
Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712

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The stability of Beltrami flows has been analyzed. The model equation represents the coupling of the Kelvin–Helmholtz (KH) instability with Alfvén waves. In a single Beltrami flow that parallels a force-free magnetic field, the magnetic field reduces the growth rate of the KH instability, while the marginally stable wave number is unchanged. Calculating the marginally stable eigenfunction of a magnetohydrodynamic flow, the necessary and sufficient condition for the exponential stability has been derived. The stability of double Beltrami flows has also been analyzed, which is represented by linear combinations of two Beltrami flows. Coupling of two vortices yields both stabilizing and destabilizing effects depending on the amplitudes and the eigenvalues of two Beltrami functions.


I. INTRODUCTION

Self-organization of ordered structure occurs in various plasma systems, in the universe as well as in laboratories. The Beltrami fields, eigenfunctions of the curl operator, describe the essential characteristics of the structures created through nonlinear field-flow couplings. The Taylor relaxed state is the most remarked model of self-organized magnetic field; the determining equation is \( \nabla \times \mathbf{B} = \lambda \mathbf{A} \) (\( \lambda \) is a real number). This Beltrami magnetic field is “force free” because the current \( \mathbf{J} = c \mathbf{B} \) (\( c \) is a real number) and \( \nabla \times \mathbf{V} = \lambda \mathbf{V} \). This field is no longer force free because of the dynamic pressure of \( \mathbf{V} \). These Beltrami magnetic and flow fields can be characterized by variational principles. The minimizer of the magnetic energy \( \int |\mathbf{B}|^2 \, dx \) (integral is taken over the total volume) for a given magnetic helicity \( \int \mathbf{A} \cdot \mathbf{B} \, dx \) (\( \mathbf{A} \) is the vector potential) is the Beltrami magnetic field. To implement a flow, we minimize \( \int (|\mathbf{V}|^2 + |\mathbf{B}|^2) \, dx \) with restricting the magnetic helicity and the cross helicity \( \int \mathbf{V} \cdot \mathbf{B} \, dx \). The Beltrami fields in the two-fluid (Hall) magnetohydrodynamic (MHD) theory span a far richer set of relaxed states—\( \mathbf{B} \) and \( \mathbf{V} \) are represented by the linear combination of the Beltrami fields. In such a “double Beltrami field,” the flow \( \mathbf{V} \) no longer parallels \( \mathbf{B} \), and the model can capture remarkably new physical effects induced by the flow.

In this paper, we study the stability of single and double Beltrami flows. We remark that the notion of “relaxed state” does not warrant the stability. Stability of a state may be proved if the kinetic part of an appropriate total energy can be shown to be bounded. If the “energy” (a constant of motion) can be split into well-defined kinetic and “potential” parts, the state with the minimum potential energy is guaranteed to be stable. For an equilibrium with a stationary flow, however, the interaction of a fluctuation and the ambient flow may not be written in a form of a potential force. Hence, the analysis of the stability is rather complicated.

The model equation represents the coupling of Kelvin–Helmholtz (KH) instability with Alfvén waves (Sec. II). A sheared magnetic field may bring about two different effects on the stability of a shear flow; one is a strong stabilization effect for sub-Alfvénic flows, and the other is the opposite destabilizing effect. The stability of single Beltrami fields is analyzed in Sec. III. We derive the necessary and sufficient condition of the stability by extending the theory of marginally stable eigenfunction in a neutral fluid. In Sec. IV, we study the stability of double Beltrami fields within the framework of the standard MHD equations.

II. FORMULATION OF THE STABILITY PROBLEM

An MHD plasma obeys the momentum and induction equations:

\[
\frac{\partial \mathbf{V}}{\partial t} = \nabla \times (\nabla \times \mathbf{V}) + \frac{1}{\rho} \left( \nabla \times \mathbf{B} \right) \times \mathbf{B} - \nabla \left( \frac{\mathbf{V}^2}{2} + p \right),
\]

(1)

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}),
\]

(2)

where \( \mathbf{B} \) is the magnetic field, \( \mathbf{V} \) is the flow velocity (we assume incompressible flows), and \( p \) is the pressure. We have normalized the magnetic field \( \mathbf{B} \) and the flow velocity \( \mathbf{V} \) by their representative values \( B^* \) and \( V^* \), scale length by the system size \( L \), time scale by \( L/V^* \), and pressure by \( \rho V^* \). The Alfvén Mach number \( a = V^*/V_A \) scales the flow velocity in the unit of the Alfvén velocity \( V_A = B^*/\sqrt{\mu_0 \rho} \) (ion mass density \( \rho \) is assumed to be constant). When \( B^* = 1/a^2 = 0 \), (1) reduces into the momentum equation of a neutral fluid.

4Present address: Plasma Physics Laboratory, University of Saskatchewan, 116 Science Place, Saskatoon, Saskatchewan S7N 5E2, Canada; electronic mail: ito@plasma.usask.ca
We consider a slab geometry in Cartesian coordinates \(x,y,z\), and assume \(\partial_z = 0\). Using a flux function \(\psi\) and a stream function \(\phi\), we may write

\[
\mathbf{B} = \nabla \psi(x,y) \times \nabla z + B_z(x,y) \nabla z, \\
\mathbf{V} = \nabla \phi(x,y) \times \nabla z + V_z(x,y) \nabla z.
\]

Equations (1) and (2) can be cast in a form of coupled non-linear Liouville equations;

\[
\partial_t(-\Delta \psi) + \{\psi,(-\Delta \phi)\} + \frac{1}{a^2}\{\psi, \Delta \phi\} = 0, \\
\partial_t \psi + \{\psi, \phi\} = 0, \\
\partial_t V_z + \{\psi, V_z\} + \frac{1}{a^2}\{B_z, \psi\} = 0, \\
\partial_t B_z + \{\phi, B_z\} + \{V_z, \psi\} = 0.
\]

Here, \(\{P,Q\} = (\partial_t P) \cdot (\partial_x Q) - (\partial_x P) \cdot (\partial_t Q)\) is the standard Poisson bracket.

We assume that the equilibrium fields \(B_0\) and \(V_0\) have only \(y\) and \(z\) components that are functions of only \(x\):

\[
B_0(x) = \{0, B_y(x), B_z(x)\}, \\
V_0(x) = \{0, V_y(x), V_z(x)\}.
\]

The thickness of the slab geometry is the system size \(L\). We consider the region of \(x\) in the interval \((0,1)\). We note that the pressure term \(\nabla \cdot p\) in Eq. (1) has been eliminated in Eq. (3) by taking the curl derivative. In a one dimensional system, the de-curl of Eq. (3) can always reproduce the pressure \(p\) by integrating \(\partial_y p\) (however, it is not so in multi dimension systems).

We consider perturbations (denoted by suffix “\(1\)” of the form of \(f_1 = f_1(x) \exp(iky - \omega t)\). Linearizing Eqs. (3)–(6) yields

\[
\Omega (\psi''_1 - k^2 \psi_1) + k V''_y \psi_1 - \frac{1}{a^2} k B^2_x \psi_1 - B_y (\psi''_1 - k^2 \psi_1) = 0, \\
\Omega \psi_1 + k B_z \psi_1 = 0, \\
\Omega B_{z1} + k B_y V_{z1} = k B^*_y \psi_1 - k V'_y \psi_1, \\
\Omega V_{z1} + \frac{1}{a^2} k B_y B_{z1} = k V'_y \psi_1 - \frac{1}{a^2} k B^*_y \psi_1,
\]

where \(\Omega = \omega - k V_y\) is the Doppler-shifted frequency. We assume rigid boundaries \((u_1 = ik \psi_1 = 0)\) at \(x = 0,1\).

Equations (7) and (8) constitute a closed system determining \(\psi_1\) and \(\phi_1\). On the other hand, Eqs. (9) and (10) include \(\psi_1, \psi_1, V_{z1}\) and \(B_{z1}\). Rewriting Eqs. (9) and (10) as

\[
\Omega \left( \Omega^2 - \frac{1}{a^2} k^2 B^2_y \right) B_{z1} - k B^*_y \left( \Omega^2 - \frac{1}{a^2} k^2 B^2_y \right) \psi_1, \\
\Omega \left( \Omega^2 - \frac{1}{a^2} k^2 B^2_y \right) V_{z1} = k V'_y \left( \Omega^2 - \frac{1}{a^2} k^2 B^2_y \right) \psi_1,
\]

we find that \(B_{z1}\) and \(V_{z1}\) describe a forced Alfvén wave with inhomogeneous driving terms (right-hand sides). Indeed, the general solutions of Eqs. (11) and (12) may be written in the form of

\[
B_{z1} = \frac{k B^*_y}{\Omega} \psi_1 + B_{1h}, \\
V_{z1} = \frac{k V'_y}{\Omega} \psi_1 + V_{1h},
\]

where \(B_{1h}\) and \(V_{1h}\) are the usual Alfvén-wave solution of the homogeneous parts of Eqs. (11) and (12). Hence, our primary interest is solving Eqs. (7) and (8) for \(\psi_1\) and \(\phi_1\).

Eliminating \(\psi_1\) from Eqs. (7) and (8), we obtain a second-order ordinary differential equation governing \(\phi_1\):

\[
\frac{d}{dx} \left[ \left( \Omega^2 - \frac{1}{a^2} k^2 B^2_y \right) \frac{d}{dx} \frac{\phi_1}{\Omega} \right] - k^2 \left( \Omega^2 - \frac{1}{a^2} k^2 B^2_y \right) \frac{\phi_1}{\Omega} = 0.
\]

When the ambient flow \(V_y\) vanishes, Eq. (15) reduces into the standard Alfvén wave equation that gives only the Alfvén continuous spectrum.10 A nonconstant \(V_y\) destroys the self-adjointness of Eq. (15)—an essential departure from the conventional Hermitian MHD.

We may rewrite Eq. (15) as

\[
(\omega - k V_y) (\phi''_1 - k^2 \phi_1) + k V_y'' \phi_1 - 2 k^2 B_y (B'_y \Omega + k B_y V'_y) \\
- \frac{k}{a^2} \Omega (\Omega^2 - k^2 B^2_y) \phi''_1 = 0.
\]

When the ambient magnetic field \(B_y\) vanishes, Eq. (16) reads as well-known Rayleigh’s equation of neutral fluid. The non-self-adjointness originates from the term including \(V''_y\). The change of the sign of \(V''_y\) may produce the KH instability (Rayleigh’s inflection-point theorem11). The third term on the left-hand side of Eq. (16) represents the effect of a magnetic field on the KH instabilities.

Equation (15) may have unstable point spectra in addition to the Alfvén continuous spectrum. Multiplying Eq. (15) by the complex conjugate of \((\phi_1/\Omega)\) and integrating over \((0,1)\), we obtain a quadratic form:

\[
\int_0^1 \left( \Omega^2 - \frac{1}{a^2} k^2 B^2_y \right) |\phi|^2 \, dx = 0, \quad (17)
\]

where

\[
|\phi|^2 = \left| \frac{d}{dx} \left( \frac{\phi_1}{\Omega} \right) \right|^2 + \frac{k^2}{a^2} \frac{\phi_1^2}{\Omega^2}.
\]

The imaginary part of Eq. (17) implies

\[
\text{Im}(\omega) \int_0^1 \{\text{Re}(\omega) - k V_y\} |\phi|^2 \, dx = 0. \quad (18)
\]

If \(\text{Im}(\omega) \neq 0\), the integrand of Eq. (18) must change the sign in \((0,1)\), i.e.,

\[
V_{y\text{min}} < \text{Re}(\omega)/k < V_{y\text{max}}.
\]
This is the reproduction of the standard relation for neutral fluids.\textsuperscript{12} The region of \( \text{Re}(\omega) \) for unstable eigenvalues is not affected by magnetic fields. A “sufficient” condition of stability can be easily found if we rewrite Eq. (17) as\textsuperscript{6,7}

\[ A \tilde{\omega}^2 + 2B \tilde{\omega} + C = 0, \]

where

\[ \tilde{\omega} = \omega - kV_0, \quad V_0 = \text{const}, \]

\[ A = \int_0^1 |\phi|^2 \, dx > 0, \]

\[ B = -k \int_0^1 \tilde{V}_y(x) |\phi|^2 \, dx, \]

\[ C = k^2 \int_0^1 \left( \tilde{V}_y(x)^2 - \frac{1}{a^2} B_y(x)^2 \right) |\phi|^2 \, dx, \]

\[ \tilde{V}_y(x) = V_y(x) - V_0. \]

If

\[ \frac{1}{a^2} B_y(x)^2 \geq \tilde{V}_y(x)^2, \]

in the whole region, the frequency in an appropriate reference frame \( \tilde{\omega} \) must be real numbers, implying stability.

It may be useful to invoke the analogy of Eq. (16) and the equation of diocotron modes in a sheared magnetic field.\textsuperscript{13} A diocotron mode is essentially a KH instability of electrostatic oscillations in a single-species plasma. When a non-neutral plasma is confined in a sheared magnetic field, the diocotron mode (electrostatic mode perpendicular to the magnetic field) is coupled with parallel electron oscillations. The latter neutralizes the charge of the diocotron mode, and hence, the mode is strongly stabilized in a sheared magnetic field. This stabilization effect appears as an additional term to Rayleigh’s equations, which is similar to the third term in Eq. (16). Here, the Alfvén waves work to reduce the instability energy.

### III. STABILITY OF BELTRAMI FLOWS

In this section, we study the stability of a Beltrami field given by

\[ \nabla \times B_0 = \lambda B_0, \]

\[ B_0 = V_0, \]

\[ \frac{V_0}{2} + p_0 = \text{const}, \]

where \( \lambda \) is a real number (the magnitude of \( V_0 \) is already normalized). In the slab geometry, we can write

\[ B_0 = V_0 = \begin{cases} 0 & \text{if } x = 0, \\ \sin(\lambda x + \delta) & \text{if } 0 < x < 1, \\ \cos(\lambda x + \delta) & \text{if } 1 < x < L. \end{cases} \]

Since \( V_y = B_y \), the sufficient condition of stability (21) reads as

\[ 1/\lambda \approx 1, \]

which implies sub-Alfvénic velocity.

The necessary and sufficient condition of stability can be obtained by searching the marginally stable eigenfunction. We extend the method of Tollmien\textsuperscript{14} that was developed for a neutral fluid. Let us start by reviewing the case of \( 1/\lambda = 0 \) (neutral fluid). KH instability is a global mode that occurs only for a finite wave number less than \( k_s \), the marginally stable wave number (Fig. 1). Tollmien has shown the existence of a marginally stable eigenfunction \( \varphi = \varphi_s \) satisfying \( \omega/\kappa = V_y(x_s) \), where \( x_s \) is an inflection point [\( V''_y(x_s) = 0 \)]. The eigenfunction for the sinusoidal flow \( V_y = \sin(\lambda x + \delta) \) (0 \( \leq x \leq 1 \)) satisfies

\[ \sin(\lambda x + \delta)\{\varphi''_s + (\lambda^2 - k^2_s)\varphi_s\} = 0, \]

where \( \varphi_s = 0 \) at \( x = 0 \) and 1. The factor \( \sin(\lambda x + \delta) \) on the left hand side of Eq. (27) produces a real continuous spectrum representing convective transport.\textsuperscript{15} The KH mode, which can become unstable, is characterized by the second factor \( \{\varphi''_s + (\lambda^2 - k^2_s)\varphi_s\} \) in Eq. (27), which yields

\[ \varphi_s = \sin n \pi x, \quad k_s = \sqrt{\lambda^2 - n^2 \pi^2} \quad (n = \pm 1, \pm 2, \ldots). \]

Figure 1 shows the growth rate as a function of the wave number \( k (\lambda = 2 \pi) \). In this case, \( \omega \) is pure imaginary. The marginally stable wave number \( k_s = \sqrt{3} \pi \approx 5.44 \) is obtained from Eq. (28). Figure 2 shows the vorticity of the eigenfunction \( \varphi_1 \). The singularity at \( x = 0.5 \) disappears in the limit of \( k \to k_s \). This behavior can be understood by rewriting Rayleigh’s equation in terms of vorticity \( \Psi = -\Delta \varphi_1 \);
The existence of a marginally stable eigenfunction is unchanged when the magnetic field is applied (1/a ≠ 0).

In what follows, we analytically demonstrate the instability for k<k_s (k_s is the critical wave number that is identical to that of the neutral fluid; see Eq. (28)). Here, we assume 1/a<1 [if 1/a≥1, KH modes are always stable; see Eq. (26)]. Since the Beltrami condition demands V_y = B_y, Eq. (16) simplifies as (c = ω/k)

\[ (V_y - c)(\varphi''_y - k^2 \varphi_y) - V''_y \varphi_1 - \frac{2cV'_yV''_y}{(V_y - c)(a^2(V_y - c)^2 - V''_y)} \]
\[ \times [(c - V_y)\varphi'_1 + V'_y\varphi_1] = 0. \]  
(30)

The existence of a marginally stable eigenfunction \( \varphi_1 = \varphi_s \) with the critical wave number k_s (>0) is almost straightforward; Equation (30) reads

\[ V_y(\varphi''_y - k^2 \varphi_y) - V''_y \varphi_1 = 0. \]  
(31)

with the boundary conditions \( \varphi_1(0) = \varphi_1(1) = 0 \). The solution of Eq. (31) is identical to Eq. (28). Hence, there is a finite interval of the wave number (0,k_s) where, as we will show, the KH mode is unstable.

There are some different perturbation methods to study the neighborhood of the marginally stable wave number.12,14,16 Here we apply the scheme of Ref. 16. We may assume \( \varphi_s(x_s) ≠ 0 \), for n = 1 in Eq. (28). We define \( \psi(x) = \varphi_s(x)\int_0^x \varphi_s(x')^{-2} dx', \) that solves Eq. (31). This \( \psi \) satisfies also the MHD equation (16), if

\[ V_y = B_y + c_0, \]
\[ B_y(x_s) = 0, \]  
(32)

where \( c_0 \) is a real constant. Our Beltrami model satisfies Eq. (32) with \( c_0 = 0 \). We easily verify the Wronskian \( W(\varphi_1, \psi_1) = 1 \) and

\[ \psi_1(0) = -1/\psi'_1(0), \quad \psi_1(1) = -1/\psi'_1(1), \]
\[ \psi(x_s) = 0 \quad \text{and} \quad \psi'(x_s) = 1/\psi_1(x_s). \]  
(33)

For (k,c) = (k_s,0), we expand \( \varphi_1(x;k,c) \) in powers of both k-k_s and c;

\[ \varphi_1(x) = \varphi_s + \Phi_1(x)(k-k_s) + \Phi_2(x)c + \cdots. \]  
(34)

Plugging Eq. (34) into (30), we find

\[ V_y(\Phi''_y - k^2 \Phi_1) - V''_y \Phi_1 = 2k_sV_y\varphi_1, \]  
(35)
\[ V_y(\Phi''_y - k^2 \Phi_2) - V''_y \Phi_2 = \frac{2V'_y(\varphi''_y - V''_y) + (a^2 - 1)V''_y}{V'_y}. \]  
(36)

The effect of the magnetic field (1/a ≠ 0) appears only as the second term on the right-hand side of Eq. (36). Therefore, \( \Phi_2 \) is modified by the magnetic field. Using \( \varphi_1 \) and \( \psi_1 \) as the Green functions, the solutions of Eqs. (35) and (36), with the boundary conditions \( \Phi_1(0) = \Phi_2(0) = 0 \), are given by

\[ \Phi_1(x) = 2k_s \left[ \int_0^x \varphi'_ys dx - \varphi'_y \int_{x_s}^x \varphi_s \psi_1 dx \right] \]  
(37)
and
\[ \Phi_2(x) = \psi_s \int_0^x \left[ \frac{V''_y}{V'_y} \varphi_x^2 + \frac{2V'_y(V_y\varphi_x - V_y\varphi_x')}{(a^2 - 1)V'_y} \varphi_x \right] \, dz \]
\[ - \varphi_x \int_x^V \frac{V''_y}{V'_y} \varphi_x \psi_s + \frac{2V'_y(V_y\varphi_x - V_y\varphi_x')}{(a^2 - 1)V'_y} \psi_s \, dz. \]

(38)

In Eq. (38), the integral is taken along a path in the complex plane to avoid the singularity of the integrand. We consider the marginal stability as the limit approached from the instability regime \((\text{Im}(c)) > 0\), and hence, the path of integration must be taken by the analytical continuation so that \(V_y\) goes below zero. At \(x = 1\), we observe

\[ \Phi_1(1) = -\frac{2k_s}{\varphi_x'(1)} \int_0^1 \varphi_x^2 \, dx, \]
\[ \Phi_2(1) = -\frac{1}{(1 - 1/a^2)\varphi_x'(1)} \int_0^1 \varphi_x'^2 \, dz. \]

(40)

The real and imaginary parts of \(\Phi_2(1)\) are given by

\[ \Phi_{2r}(1) = -\frac{1}{(1 - 1/a^2)\varphi_x'(1)} \int_0^1 \varphi_x'^2 \, dx \]
and

\[ \Phi_{2i}(1) = -\frac{\pi}{(1 - 1/a^2)\varphi_x'^2(1)} \int_0^1 \frac{V''_y}{V'_y} \varphi_x'^2(1) \, dx \]
\[ \times \text{sgn}V'_y(x_s). \]

(42)

where \(P\) denotes the Cauchy principal value of the integral. By the definitions (37) and (38), \(\varphi_1\) of Eq. (34) vanishes at \(x = 0\). The other boundary condition \(\varphi_1(1) = 0\) demands

\[ c = -\frac{\Phi_1(1)\Phi_2'(1)}{[\Phi_2(1)]^2} (k - k_s). \]

(43)

The imaginary part of Eq. (43) reads

\[ \text{Im}(c) = \frac{\Phi_1(1)\Phi_2'(1)}{[\Phi_2(1)]^2} (k - k_s) \]
\[ = C_1(1 - 1/a^2) \quad (1/a \leq 1), \]

(45)

where \(C_1 = \text{Im}(c)\) for \(1/a = 0\). Since \(V''_y(x_s)\text{sgn}V'_y(x_s) < 0\) for the Beltrami flow, Eq. (44) shows that \(\text{Im}(c)\) is positive for \(k < k_s\) when \(1/a < 1\). Equation (45) shows that the growth rate decreases when the magnetic field (\(\propto 1/a\)) increases.

In Fig. 3, we compare Eq. (45) with the numerical result. Combining with the abovementioned sufficient condition of stability, the necessary and sufficient condition for stability of the Beltrami flow is \(\lambda \leq \pi\) or \(1/a \geq 1\) (see Fig. 5).

IV. STABILITY OF DOUBLE BELTRAMI FLOWS

In this section, we analyze the stability of double Beltrami flows. We use the Alfvén unit to set \(1/a_0 = 1\). The double Beltrami fields are represented by linear combinations of two Beltrami flows:

\[ B_0 = C_1 G_1 + C_2 G_2, \]

(46)

\[ V_0 = D_1 G_1 + D_2 G_2, \]

(47)

where

\[ \nabla \times G_j = \lambda_j G_j \quad (j = 1, 2), \]
\[ D_1/C_1 = (\lambda_1 + \alpha_{-1}), \]
\[ D_2/C_2 = (\lambda_2 + \alpha_{-1}), \]
\[ \alpha_{\pm} = 2\left[ -\left(\lambda_1 + \lambda_2\right) \pm \sqrt{\left(\lambda_1 - \lambda_2\right)^2 + 4}\right]^{-1}. \]

(48)

(49)

(50)

We can consider two different pairs of amplitudes (48) and (49) depending on the choice of the sign of \(\alpha_{\pm}\). In the slab geometry, Beltrami fields are given by

\[ G_j = (0, \sin \lambda_j x, \cos \lambda_j x) \quad (0 \leq x < 1, j = 1, 2). \]

The necessary and sufficient condition for the stability of each separate Beltrami vortex is (see Sec. III)

\[ |D_j/C_j| \leq 1 \quad \text{or} \quad \lambda_j \leq \pi \quad (j = 1, 2). \]

We present our results in a parameter space of \(\lambda\) (reciprocal length scale) and \(1/a\) (magnetic field strength). The stability condition on the axis \(1/a = 0\) is consistent with the well-known result of the KH mode in neutral fluids (Ref. 14). The stable region of \(1/a \ll 1\) was predicted by the quadratic form argument (Ref. 6).

![Fig. 5](image)

FIG. 5. The stable and unstable regions in the parameter space of \(\lambda\) (reciprocal length scale) and \(1/a\) (magnetic field strength). The stability condition on the axis \(1/a = 0\) is consistent with the well-known result of the KH mode in neutral fluids (Ref. 14). The stable region of \(1/a \ll 1\) was predicted by the quadratic form argument (Ref. 6).

\[ V_0 = D_1 G_1 + D_2 G_2, \]

(47)

where

\[ \nabla \times G_j = \lambda_j G_j \quad (j = 1, 2), \]
\[ D_1/C_1 = (\lambda_1 + \alpha_{-1}), \]
\[ D_2/C_2 = (\lambda_2 + \alpha_{-1}), \]
\[ \alpha_{\pm} = 2\left[ -\left(\lambda_1 + \lambda_2\right) \pm \sqrt{\left(\lambda_1 - \lambda_2\right)^2 + 4}\right]^{-1}. \]

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The necessary and sufficient condition for the stability of each separate Beltrami vortex is (see Sec. III)

\[ |D_j/C_j| \leq 1 \quad \text{or} \quad \lambda_j \leq \pi \quad (j = 1, 2). \]
Equations (48) and (49) show that the values \( D_j / C_j \) \((j = 1,2)\) are defined by \( \lambda_1 \) and \( \lambda_2 \), and their product must satisfy

\[
(D_1 / C_1)(D_2 / C_2) = 1,
\]

which means that one vortex is sub-Alfvénic while the other is super-Alfvénic \((|D_1 / C_1| = |D_2 / C_2| = 1\) occurs when \( \lambda_1 = \lambda_2 \). A combination of two unstable vortices is not possible in a slab geometry. There are two types of combinations:

(A) super-Alfvénic vortex with smaller structure and sub-Alfvénic vortex with larger structure \((|D_1 / C_1| > 1, |D_2 / C_2| < 1 \) and \( \lambda_1 > \lambda_2 \)).

(B) super-Alfvénic vortex with larger structure and sub-Alfvénic vortex with smaller structure \((|D_1 / C_1| > 1, |D_2 / C_2| < 1 \) and \( \lambda_1 < \lambda_2 \)).

We distinguish the super- and sub-Alfvénic vortices by subscripts 1 and 2, respectively. In the double Beltrami fields, the profiles of \( B_y \) and \( V_y \) do not satisfy Eq. (32), and hence, the marginally stable eigenfunction cannot be found. In the case (A) with \( \lambda_1 = n \lambda_2 \) \((n \) is an integer number) or \( \lambda_2 \approx \pi \), the sufficient condition for the stability, \( B^2_1 \geq V^2_1 \) in the whole region, holds for some appropriate choices of \( C_1 \) and \( C_2 \). Otherwise, however, the sufficient condition is satisfied only at the limit of \( |C_2 / C_1| \to \infty \).

Fixing \( C_1 = 1 \) and taking \( C_2 \) as a control parameter, we compare the growth rates of Cases (A) and (B) for the same pair of \( \lambda_1 \) and \( \lambda_2 \). We denote the larger (absolute value) one of \( \lambda_{1,2} \) by \( \lambda_1 \) (smaller size) and the smaller one by \( \lambda_1 \). For given \( \lambda_1 \) and \( \lambda_1 \), the selection of \( \alpha_+ \) or \( \alpha_- \) in Eqs. (48) and (49) switches the Cases (A) and (B).

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**FIG. 7.** The profiles of \( B_y \), \( V_y \), \( B^2_1 \), \( V^2_1 \) and \( V\gamma \) versus \( C_2 \) in Case (A) of Fig. 6.

**FIG. 8.** The profiles of \( B_y \), \( V_y \), \( B^2_1 \), \( V^2_1 \) and \( V\gamma \) versus \( C_2 \) in Case (B) of Fig. 6.
Figure 6 shows the maximum growth rate as a function of $C_2$ (we choose $\lambda_s=2\pi$ and $\lambda_l=\pi$). If we use $\alpha_-$, the vortex of $\lambda_s$ is unstable and that of $\lambda_l$ is stable [Case (A)]. The profiles of $B_y$ and $V_y$ are shown in Fig. 7 as functions of $C_2$. When we increase $C_2$, the amplitude of the magnetic field increases. For $C_2 > 6.87$, the local Alfvén velocity exceeds the flow velocity everywhere in the domain $(0, 1)$. In Fig. 7, we observe that the instability is suppressed for $C_2 \geq 6.5$, implying that the criteria ($B_y^2 \approx V_y^2$ in the whole domain) is almost the necessary and sufficient condition.

If we use $\alpha_-$, Case (B) occurs. Both vortices ($\lambda_s$ and $\lambda_l$) are separately stable (the small vortex $\lambda_s$ becomes sub-Alfvénic). The profiles of $B_y$ and $V_y$ are shown in Fig. 8. The combination of stable vortices causes instability. However, the growth rate is smaller than that of Case (A) (see Fig. 6). The flow of the combined vortices does not have an inflection point if $C_2 < 1.47$ (see Fig. 8). If there was no magnetic field, this flow is stable (Rayleigh’s inflection-point theorem). However, the double Beltrami field is unstable when $C_2 \geq 1$ (see Fig. 6). This is the so-called joint instability—an example of the destabilizing effect of a magnetic field.

V. SUMMARY

The stability of a plasma with ambient flow is a rather complex problem. Both flow shear and magnetic shear are double edge sword—they have both stabilizing and destabilizing effects. The stretching effect of a shear flow is believed to stabilize instabilities. However, the evolution of fluctuations in an ambient shear flow is rather complicated. Nonexponential (possibly algebraic) behavior of fluctuations are left outside the scope of present paper. A sufficient condition of general stability (including even the non-linear regime) will be discussed elsewhere. A shear flow contains a free energy to excite KH instabilities.

The existence of marginally stable eigenfunction has been shown analytically, which provides us with the necessary and sufficient condition for the exponential stability. We have seen that the magnetic shear effect reduces the growth rate of the instability, while the critical wave number is unchanged. We note that the assumption of a slab geometry omits the kink instabilities that are induced by the magnetic field curvature effect. Instabilities in a cylindrical geometry will be discussed elsewhere. The Beltrami fields are simple in the sense of the relation (32) that allows us to construct the marginally stable solution.

In a general MHD flow, we cannot find the marginally stable solution, so that the necessary and sufficient condition of stability is not known. The complexity of the stability problem is seen in the analysis of the double Beltrami fields. We have shown that the joint instability (destabilization by a magnetic field) may occur.

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15Due to the non-Hermitian property of the system, eigenfunctions of modes are not orthogonal with each other. Hence, KH-mode may couple with other modes (including continuous spectra). Such mode couplings may bring about rather complex time-asymptotic phenomena [see M. Hirota, T. Tatsuno, S. Kondoh, and Z. Yoshida, Phys. Plasmas 9, 1177 (2002)]. However, those “secular” behavior is at most algebraic in $t$. If KH mode is unstable ($\text{Im } \omega > 0$), it dominates the long-term behavior.