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Kelvin–Helmholtz instability in Beltrami fields

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The stability of Beltrami flows has been analyzed. The model equation represents the coupling of the Kelvin–Helmholtz (KH) instability with Alfvén waves. In a single Beltrami flow that parallels a force-free magnetic field, the magnetic field reduces the growth rate of the KH instability, while the marginally stable wave number is unchanged. Calculating the marginally stable eigenfunction of a magnetohydrodynamic flow, the necessary and sufficient condition for the exponential stability has been derived. The stability of double Beltrami flows has also been analyzed, which is represented by linear combinations of two Beltrami flows. Coupling of two vortices yields both stabilizing and destabilizing effects depending on the amplitudes and the eigenvalues of two Beltrami functions.
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I. INTRODUCTION

Self-organization of ordered structure occurs in various plasma systems, in the universe as well as in laboratories. The Beltrami fields, eigenfunctions of the curl operator[1] describe the essential characteristics of the structures created through nonlinear field-flow couplings. The Taylor relaxed state[2] is the most remarked model of self-organized magnetic field; the determining equation is $\nabla\times\mathbf{B}=\lambda\mathbf{B}$ (λ is a real number). This Beltrami magnetic field is ''force free'' because the current ($\nabla\times\mathbf{B}$) parallels the magnetic field. A more general class of relaxed state may have a field-aligned flow satisfying $\mathbf{V}=c\mathbf{B}$ (c is a real number) and $\nabla\times\mathbf{V}=\lambda\mathbf{V}$. This field is no longer force free because of the dynamic pressure of $\mathbf{V}$. These Beltrami magnetic and flow fields can be characterized by variational principles.[2] The minimizer of the magnetic energy $\int|\mathbf{B}|^2\,dx$ (integral is taken over the total volume) for a given magnetic helicity $\int\mathbf{A}\cdot\mathbf{B}\,dx$ ($\mathbf{A}$ is the vector potential) is the Beltrami magnetic field. To implement a flow, we minimize $\int(|\mathbf{V}|^2+|\mathbf{B}|^2)\,dx$ with restricting the magnetic helicity and the cross helicity $\int\mathbf{V}\cdot\mathbf{B}\,dx$. The Beltrami fields in the two-fluid (Hall) magnetohydrodynamic (MHD) theory span a far richer set of relaxed states—$\mathbf{B}$ and $\mathbf{V}$ are represented by the linear combination of two Beltrami fields.[3,4] In such a ''double Beltrami field,'' the flow $\mathbf{V}$ no longer parallels $\mathbf{B}$, and the model can capture remarkably new physical effects induced by the flow.[5]

In this paper, we study the stability of single and double Beltrami flows. We remark that the notion of ''relaxed state'' does not warrant the stability. Stability of a state may be proved if the kinetic part of an appropriate total energy can be shown to be bounded. If the ''energy'' (a constant of motion) can be split into well-defined kinetic and ''potential'' parts, the state with the minimum potential energy is guaranteed to be stable. For an equilibrium with a stationary flow, however, the interaction of a fluctuation and the ambient flow may not be written in a form of a potential force. Hence, the analysis of the stability is rather complicated.

The model equation represents the coupling of Kelvin–Helmholtz (KH) instability with Alfvén waves (Sec. II). A sheared magnetic field may bring about two different effects on the stability of a shear flow; one is a strong stabilization effect for sub-Alfvénic flows,[6–8] and the other is the opposite destabilizing effect.[6,9] The stability of single Beltrami fields is analyzed in Sec. III. We derive the necessary and sufficient condition of the stability by extending the theory of marginally stable eigenfunction in a neutral fluid. In Sec. IV, we study the stability of double Beltrami fields within the framework of the standard MHD equations.

II. FORMULATION OF THE STABILITY PROBLEM

An MHD plasma obeys the momentum and induction equations;

$$\frac{\partial\mathbf{V}}{\partial t}=\mathbf{V}\times(\nabla\times\mathbf{V})+\frac{1}{a^2}(\nabla\times\mathbf{B})\times\mathbf{B}-\nabla\left(\frac{\mathbf{V}^2}{2}+p\right), \tag{1}$$

$$\frac{\partial\mathbf{B}}{\partial t}=\nabla\times(\mathbf{V}\times\mathbf{B}), \tag{2}$$

where $\mathbf{B}$ is the magnetic field, $\mathbf{V}$ is the flow velocity (we assume incompressible flows), and p is the pressure. We have normalized the magnetic field $\mathbf{B}$ and the flow velocity $\mathbf{V}$ by their representative values B^* and V^*, scale length by the system size L, time scale t by L/V^* and pressure p by ρV^{*2}. The Alfvén Mach number $a=V^*/V_A$ scales the flow velocity in the unit of the Alfvén velocity $V_A=B^*/\sqrt{\mu_0\rho}$ (ion mass density ρ is assumed to be constant). When $B^*=0$ ($1/a^2=0$), (1) reduces into the momentum equation of a neutral fluid.

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We consider a slab geometry in Cartesian coordinates x,y,z, and assume $\partial_z=0$. Using a flux function ψ and a stream function φ, we may write

$$\mathbf{B}=\nabla\psi(x,y)\times\nabla z+B_z(x,y)\nabla z,$$

$$\mathbf{V}=\nabla\varphi(x,y)\times\nabla z+V_z(x,y)\nabla z.$$

Equations (1) and (2) can be cast in a form of coupled nonlinear Liouville equations;

$$\partial_t(-\Delta\varphi)+\{\varphi,(-\Delta\varphi)\}+\frac{1}{a^2}\{\psi,\Delta\psi\}=0, \tag{3}$$

$$\partial_t\psi+\{\varphi,\psi\}=0, \tag{4}$$

$$\partial_t V_z+\{\varphi,V_z\}+\frac{1}{a^2}\{B_z,\psi\}=0, \tag{5}$$

$$\partial_t B_z+\{\varphi,B_z\}+\{V_z,\psi\}=0. \tag{6}$$

Here, $\{P,Q\}\equiv(\partial_y P)\cdot(\partial_x Q)-(\partial_x P)\cdot(\partial_y Q)$ is the standard Poisson bracket.

We assume that the equilibrium fields $\mathbf{B}_0$ and $\mathbf{V}_0$ have only y and z components that are functions of only x:

$$\mathbf{B}_0(x)=[0,B_y(x),B_z(x)],$$

$$\mathbf{V}_0(x)=[0,V_y(x),V_z(x)].$$

The thickness of the slab geometry is the system size L. We consider the region of x in the interval (0,1). We note that the pressure term ∇p in Eq. (1) has been eliminated in Eq. (3) by taking the curl derivative. In a one dimensional system, the de-curl of Eq. (3) can always reproduce the pressure p by integrating $\partial_x p$ (however, it is not so in multi dimension systems).

We consider perturbations (denoted by suffix "1") of the form of $f_1=f_1(x)\exp i(ky-\omega t)$. Linearizing Eqs. (3)–(6) yields

$$\Omega(\varphi_1''-k^2\varphi_1)+kV_y''\varphi_1-\frac{1}{a^2}k\{B_y''\psi_1-B_y(\psi_1''-k^2\psi_1)\}=0, \tag{7}$$

$$\Omega\psi_1+kB_y\varphi_1=0, \tag{8}$$

$$\Omega B_{z1}+kB_yV_{z1}=kB_z'\varphi_1-kV_z'\psi_1, \tag{9}$$

$$\Omega V_{z1}+\frac{1}{a^2}kB_yB_{z1}=kV_z'\varphi_1-\frac{1}{a^2}kB_z'\psi_1, \tag{10}$$

where $\Omega=\omega-kV_y$ is the Doppler-shifted frequency. We assume rigid boundaries ($v_x=ik\varphi_1=0$) at $x=0,1$.

Equations (7) and (8) constitute a closed system determining φ_1 and ψ_1. On the other hand, Eqs. (9) and (10) include φ_1, ψ_1, V_{z1} and B_{z1}. Rewriting Eqs. (9) and (10) as

$$\Omega\left(\Omega^2-\frac{1}{a^2}k^2B_y^2\right)B_{z1}=kB_z'\left(\Omega^2-\frac{1}{a^2}k^2B_y^2\right)\varphi_1, \tag{11}$$

$$\Omega\left(\Omega^2-\frac{1}{a^2}k^2B_y^2\right)V_{z1}=kV_z'\left(\Omega^2-\frac{1}{a^2}k^2B_y^2\right)\varphi_1, \tag{12}$$

we find that B_{z1} and V_{z1} describe a forced Alfvén wave with inhomogeneous driving terms (right-hand sides). Indeed, the general solutions of Eqs. (11) and (12) may be written in the form of

$$B_{z1}=\frac{kB_z'}{\Omega}\varphi_1+B_{z1h}, \tag{13}$$

$$V_{z1}=\frac{kV_z'}{\Omega}\varphi_1+V_{z1h}, \tag{14}$$

where B_{z1h} and V_{z1h} are the usual Alfvén-wave solution of the homogeneous parts of Eqs. (11) and (12). Hence, our primary interest is solving Eqs. (7) and (8) for φ_1 and ψ_1.

Eliminating ψ_1 from Eqs. (7) and (8), we obtain a second-order ordinary differential equation governing φ_1;

$$\frac{d}{dx}\left[\left(\Omega^2-\frac{1}{a^2}k^2B_y^2\right)\frac{d}{dx}\left(\frac{\varphi_1}{\Omega}\right)\right]-k^2\left(\Omega^2-\frac{1}{a^2}k^2B_y^2\right)\left(\frac{\varphi_1}{\Omega}\right)$$
$$=0. \tag{15}$$

When the ambient flow V_y vanishes, Eq. (15) reduces into the standard Alfvén wave equation that gives only the Alfvén continuous spectrum.[10] A nonconstant V_y destroys the self-adjointness of Eq. (15)—an essential departure from the conventional Hermitian MHD.

We may rewrite Eq. (15) as

$$(\omega-kV_y)(\varphi_1''-k^2\varphi_1)+kV_y''\varphi_1$$
$$-\frac{2k^2B_y(B_y'\Omega+kB_yV_y')}{a^2\Omega(\Omega^2-k^2B_y^2/a^2)}(\Omega\varphi_1'+kV_y'\varphi_1)=0. \tag{16}$$

When the ambient magnetic field B_y vanishes, Eq. (16) reads as well-known Rayleigh's equation of neutral fluid. The non-self-adjointness originates from the term including V_y''. The change of the sign of V_y'' may produce the KH instability (Rayleigh's inflection-point theorem[11]). The third term on the left-hand side of Eq. (16) represents the effect of a magnetic field on the KH instabilities.

Equation (15) may have unstable point spectra in addition to the Alfvén continuous spectrum. Multiplying Eq. (15) by the complex conjugate of (φ_1/Ω) and integrating over (0,1), we obtain a quadratic form;

$$\int_0^1\left(\Omega^2-\frac{1}{a^2}k^2B_y^2\right)|\phi|^2\,dx=0, \tag{17}$$

where

$$|\phi|^2=\left|\frac{d}{dx}\left(\frac{\varphi_1}{\Omega}\right)\right|^2+k^2\left|\frac{\varphi_1}{\Omega}\right|^2.$$

The imaginary part of Eq. (17) implies

$$\mathrm{Im}(\omega)\int_0^1\{\mathrm{Re}(\omega)-kV_y\}|\phi|^2\,dx=0. \tag{18}$$

If $\mathrm{Im}(\omega)\neq 0$, the integrand of Eq. (18) must change the sign in (0,1), i.e.,

$$V_{y\min}<\mathrm{Re}(\omega)/k<V_{y\max}. \tag{19}$$

This is the reproduction of the standard relation for neutral fluids.[12] The region of $\mathrm{Re}(\omega)$ for unstable eigenvalues is not affected by magnetic fields. A "sufficient" condition of stability can be easily found if we rewrite Eq. (17) as[6,7]

$$A\tilde{\omega}^2+2B\tilde{\omega}+C=0, \tag{20}$$

where

$$\tilde{\omega}=\omega-kV_0, \quad V_0=\text{const},$$

$$A=\int_0^1|\phi|^2\,dx>0,$$

$$B=-k\int_0^1\tilde{V}_y(x)|\phi|^2\,dx,$$

$$C=k^2\int_0^1\left\{\tilde{V}_y(x)^2-\frac{1}{a^2}B_y(x)^2\right\}|\phi|^2\,dx,$$

$$\tilde{V}_y(x)=V_y(x)-V_0.$$

If

$$\frac{1}{a^2}B_y(x)^2\geqslant\tilde{V}_y(x)^2, \tag{21}$$

in the whole region, the frequency in an appropriate reference frame $\tilde{\omega}$ must be real numbers, implying stability.

It may be useful to invoke the analogy of Eq. (16) and the equation of diocotron modes in a sheared magnetic field.[13] A diocotron mode is essentially a KH instability of electrostatic oscillations in a single-species plasma. When a non-neutral plasma is confined in a sheared magnetic field, the diocotron mode (electrostatic mode perpendicular to the magnetic field) is coupled with parallel electron oscillations. The latter neutralizes the charge of the diocotron mode, and hence, the mode is strongly stabilized in a sheared magnetic field. This stabilization effect appears as an additional term to Rayleigh's equations, which is similar to the third term in Eq. (16). Here, the Alfvén wave works to reduce the instability energy.

III. STABILITY OF BELTRAMI FLOWS

In this section, we study the stability of a Beltrami field given by

$$\nabla\times\mathbf{B}_0=\lambda\mathbf{B}_0, \tag{22}$$

$$\mathbf{B}_0=\mathbf{V}_0, \tag{23}$$

$$\frac{V_0^2}{2}+p_0=\text{const}, \tag{24}$$

where λ is a real number (the magnitude of $\mathbf{V}_0$ is already normalized). In the slab geometry, we can write

$$\mathbf{B}_0=\mathbf{V}_0=\begin{pmatrix}0\\ \sin(\lambda x+\delta)\\ \cos(\lambda x+\delta)\end{pmatrix}, \quad (0\leqslant x\leqslant 1). \tag{25}$$

Since $V_y=B_y$, the sufficient condition of stability (21) reads as

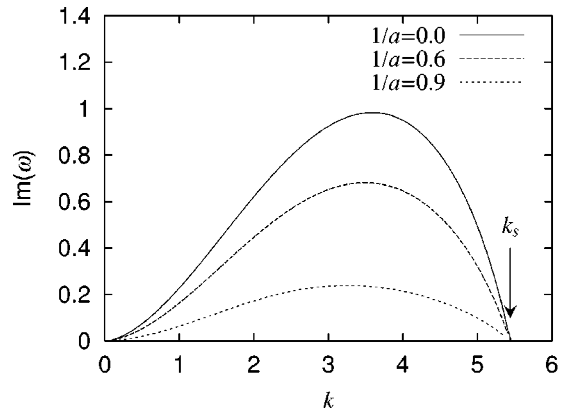

FIG. 1. The growth rate versus the wave number for three different parameters $1/a$ with $\lambda=2\pi$ and $\delta=0$.

$$1/a\geqslant 1, \tag{26}$$

which implies sub-Alfvénic velocity.

The necessary and sufficient condition of stability can be obtained by searching the marginally stable eigenfunction. We extend the method of Tollmien[14] that was developed for a neutral fluid. Let us start by reviewing the case of $1/a=0$ (neutral fluid). KH instability is a global mode that occurs only for a finite wave number less than k_s, the marginally stable wave number (Fig. 1). Tollmien has shown the existence of a marginally stable eigenfunction $\varphi=\varphi_s$ satisfying $\omega/k_s=V_y(x_s)$, where x_s is an inflection point $[V_y''(x_s)=0]$. The eigenfunction for the sinusoidal flow $V_y=\sin(\lambda x+\delta)$ $(0\leqslant x\leqslant 1)$ satisfies

$$\sin(\lambda x+\delta)\{\varphi_s''+(\lambda^2-k_s^2)\varphi_s\}=0, \tag{27}$$

where $\varphi_s=0$ at $x=0$ and 1. The factor $\sin(\lambda x+\delta)$ on the left hand side of Eq. (27) produces a real continuous spectrum representing convective transport.[15] The KH mode, which can become unstable, is characterized by the second factor $\{\varphi_s''+(\lambda^2-k_s^2)\varphi_s\}$ in Eq. (27), which yields

$$\varphi_s=\sin n\pi x, \quad k_s=\sqrt{\lambda^2-n^2\pi^2} \quad (n=\pm 1,\pm 2,\ldots). \tag{28}$$

Figure 1 shows the growth rate as a function of the wave number k $(\lambda=2\pi)$. In this case, ω is pure imaginary. The marginally stable wave number $k_s=\sqrt{3}\pi\simeq 5.44$ is obtained from Eq. (28). Figure 2 shows the vorticity of the eigenfunction φ_1. The singularity at $x=0.5$ disappears in the limit of $k\to k_s$. This behavior can be understood by rewriting Rayleigh's equation in terms of vorticity $\Psi=-\Delta\varphi_1$;

$$(V_y-c)\Psi=V_y''K\Psi, \tag{29}$$

where $c=\omega/k$ and K denotes the inverse-Laplacian operator. If c is a complex number, $\Psi(x_s)$ is zero, while if $c=V_y(x_s)$, we have $\Psi(x_s)\neq 0$. The stability condition is obtained from Eq. (28). If $\lambda\leqslant\pi$, the flow is stable even when V_y has an inflection point. If $\lambda>\pi$, the flow is always unstable.

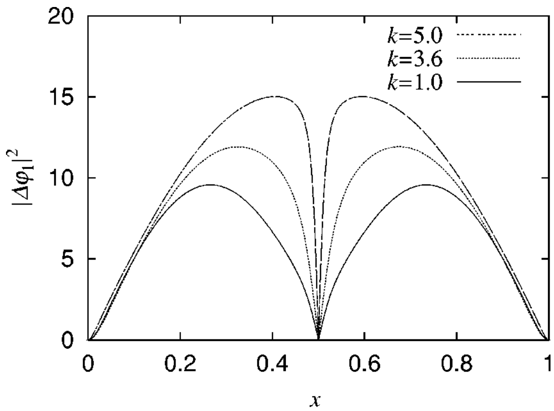

FIG. 2. The vorticity of the eigenfunction φ_1 for three different values of k, where $1/a=0$, $\lambda=2\pi$ and $\delta=0$.

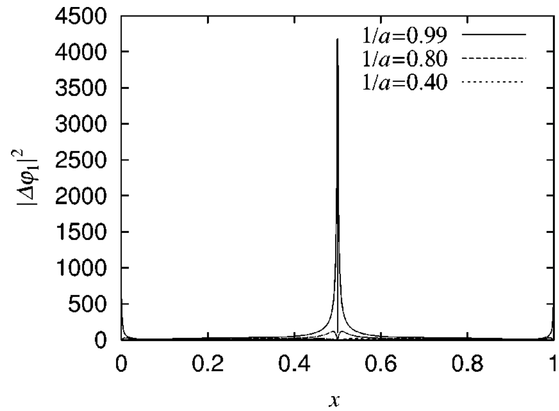

FIG. 4. The vorticity of the eigenfunction φ_1 for three different values of $1/a$. $k=3.6$, $\lambda=2\pi$ and $\delta=0$.

When we increase the magnetic field $(1/a)$, the growth rate diminishes (Fig. 3). The eigenfunction φ_1 tends to be singular near stable regions (Fig. 4), reflecting the essential singularity of the Alfvén continuum.

From this numerical analysis, we observe that the critical wave number k_s is unchanged when the magnetic field is applied $(1/a \neq 0)$.

In what follows, we analytically demonstrate the instability for $k<k_s$ [k_s is the critical wave number that is identical to that of the neutral fluid; see Eq. (28)]. Here, we assume $1/a<1$ [if $1/a \geqslant 1$, KH modes are always stable; see Eq. (26)]. Since the Beltrami condition demands $V_y=B_y$, Eq. (16) simplifies as $(c=\omega/k)$

$$(V_y-c)(\varphi_1''-k^2\varphi_1)-V_y''\varphi_1-\frac{2cV_yV_y'}{(V_y-c)\{a^2(V_y-c)^2-V_y^2\}}\times\{(c-V_y)\varphi_1'+V_y'\varphi_1\}=0. \tag{30}$$

The existence of a marginally stable eigenfunction $\varphi_1=\varphi_s$ with the critical wave number k_s (>0) is almost straightforward; Equation (30) reads

$$V_y(\varphi_s''-k_s^2\varphi_s)-V_y''\varphi_s=0, \tag{31}$$

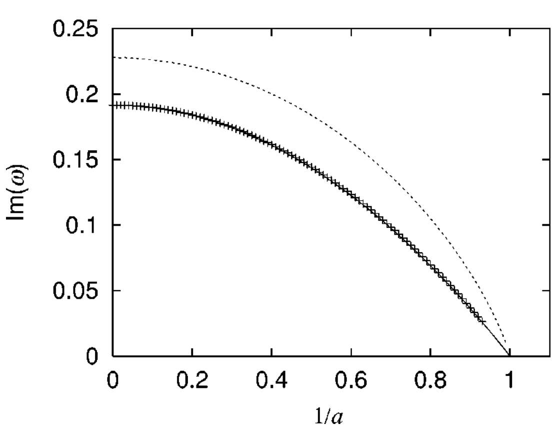

FIG. 3. The growth rate Im(ω) as functions of the parameter $1/a$. The dashed line is for $\lambda=2\pi$, $\delta=0$ and $k=1$. The points $\lambda=2\pi$, $\delta=0$ and $k=5.3$ and the solid line is the analytic curve (45).

with the boundary conditions $\varphi_s(0)=\varphi_s(1)=0$. The solution of Eq. (31) is identical to Eq. (28). Hence, there is a finite interval of the wave number $(0,k_s)$ where, as we will show, the KH mode is unstable.

There are some different perturbation methods to study the neighborhood of the marginally stable wave number.[12,14,16] Here we apply the scheme of Ref. 16. We may assume $\varphi_s(x_s)\neq 0$, for $n=1$ in Eq. (28). We define $\psi_s(x)=\varphi_s(x)\int_{x_s}^{x}\{\varphi_s(x)\}^{-2}\,dx$, that solves Eq. (31). This ψ_s satisfies also the MHD equation (16), if

$$\begin{aligned} &V_y=B_y+c_0,\\ &B_y(x_s)=0, \end{aligned} \tag{32}$$

where c_0 is a real constant. Our Beltrami model satisfies Eq. (32) with $c_0=0$. We easily verify the Wronskian $W(\varphi_s,\psi_s)=1$ and

$$\begin{aligned} &\psi_s(0)=-1/\varphi_s'(0), \quad \psi_s(1)=-1/\varphi_s'(1),\\ &\psi(x_s)=0 \quad \text{and} \quad \psi'(x_s)=1/\varphi_s(x_s). \end{aligned} \tag{33}$$

For $(k,c)\approx(k_s,0)$, we expand $\varphi_1(x;k,c)$ in powers of both $k-k_s$ and c;

$$\varphi_1(x)=\varphi_s+\Phi_1(x)(k-k_s)+\Phi_2(x)c+\cdots. \tag{34}$$

Plugging Eq. (34) into (30), we find

$$V_y(\Phi_1''-k_s^2\Phi_1)-V_y''\Phi_1=2k_sV_y\varphi_s, \tag{35}$$

$$V_y(\Phi_2''-k_s^2\Phi_2)-V_y''\Phi_2=\frac{V_y''}{V_y}\varphi_s+\frac{2V_y'(V_y'\varphi_s-V_y\varphi_s')}{(a^2-1)V_y^2}. \tag{36}$$

The effect of the magnetic field $(1/a\neq 0)$ appears only as the second term on the right-hand side of Eq. (36). Therefore, Φ_2 is modified by the magnetic field. Using ϕ_s and ψ_s as the Green functions, the solutions of Eqs. (35) and (36), with the boundary conditions $\Phi_1(0)=\Phi_2(0)=0$, are given by

$$\Phi_1(x)=2k_s\left(\psi_s\int_0^x\varphi_s^2\,dx-\varphi_s\int_{x_s}^x\varphi_s\psi_s\,dx\right) \tag{37}$$

and

$$\Phi_2(x)=\psi_s\int_0^x\left[\frac{V_y''}{V_y^2}\varphi_s^2+\frac{2V_y'(V_y'\varphi_s-V_y\varphi_s')}{(a^2-1)V_y^3}\varphi_s\right]dz$$
$$-\varphi_s\int_{x_s}^x\left[\frac{V_y''}{V_y^2}\varphi_s\psi_s+\frac{2V_y'(V_y'\varphi_s-V_y\varphi_s')}{(a^2-1)V_y^3}\psi_s\right]dz. \tag{38}$$

In Eq. (38), the integral is taken along a path in the complex plane to avoid the singularity of the integrand. We consider the marginal stability as the limit approached from the instability regime ($\mathrm{Im}(c)>0$), and hence, the path of integration must be taken by the analytical continuation so that V_y goes below zero. At $x=1$, we observe

$$\Phi_1(1)=-\frac{2k_s}{\varphi_s'(1)}\int_0^1\varphi_s^2\,dx, \tag{39}$$

$$\Phi_2(1)=-\frac{1}{(1-1/a^2)\varphi_s'(1)}\int_0^1\frac{V_y''}{V_y^2}\varphi_s^2\,dz. \tag{40}$$

The real and imaginary parts of $\Phi_2(1)$ are given by

$$\Phi_{2r}(1)=-\frac{1}{(1-1/a^2)\varphi_s'(1)}P\int_0^1\frac{V_y''}{V_y^2}\varphi_s^2\,dx \tag{41}$$

and

$$\Phi_{2i}(1)=-\frac{\pi}{(1-1/a^2)}\frac{V_y'''(x_s)\,\varphi_s^2(x_s)}{V_y'^2(x_s)\,\varphi_s'(1)}\mathrm{sgn}V_y'(x_s), \tag{42}$$

where P denotes the Cauchy principal value of the integral. By the definitions (37) and (38), φ_1 of Eq. (34) vanishes at $x=0$. The other boundary condition $\varphi_1(1)=0$ demands

$$c=-\frac{\Phi_1(1)\Phi_2^*(1)}{|\Phi_2(1)|^2}(k-k_s). \tag{43}$$

The imaginary part of Eq. (43) reads

$$\mathrm{Im}(c)=\frac{\Phi_1(1)\Phi_{2i}(1)}{|\Phi_2(1)|^2}(k-k_s) \tag{44}$$

$$=C_i(1-1/a^2)\quad(1/a<1), \tag{45}$$

where $C_i=\mathrm{Im}(c)$ for $1/a=0$. Since $V_y'''(x_s)\mathrm{sgn}\,V_y'(x_s)<0$ for the Beltrami flow, Eq. (44) shows that $\mathrm{Im}(c)$ is positive for $k\lesssim k_s$ when $1/a<1$. Equation (45) shows that the growth rate decreases when the magnetic field ($\propto 1/a$) increases.

In Fig. 3, we compare Eq. (45) with the numerical result. Combining with the abovementioned sufficient condition of stability, the necessary and sufficient condition for stability of the Beltrami flow is $\lambda\leqslant\pi$ or $1/a\geqslant 1$ (see Fig. 5).

IV. STABILITY OF DOUBLE BELTRAMI FLOWS

In this section, we analyze the stability of double Beltrami flows. We use the Alfvén unit to set $1/a=1$. The double Beltrami fields are represented by linear combinations of two Beltrami flows:[3]

$$\mathbf{B}_0=C_1\mathbf{G}_1+C_2\mathbf{G}_2, \tag{46}$$

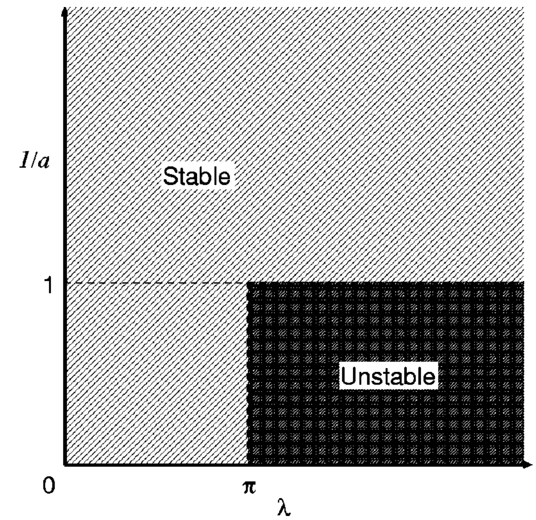

FIG. 5. The stable and unstable regions in the parameter space of λ (reciprocal length scale) and $1/a$ (magnetic field strength). The stability condition on the axis $1/a=0$ is consistent to the well-known result of the KH mode in neutral fluids (Ref. 14). The stable region of $1/a\leqslant 1$ was predicted by the quadratic form argument (Ref. 6).

$$\mathbf{V}_0=D_1\mathbf{G}_1+D_2\mathbf{G}_2, \tag{47}$$

where

$$\nabla\times\mathbf{G}_j=\lambda_j\mathbf{G}_j\quad(j=1,2),$$
$$D_1/C_1=(\lambda_1+\alpha_\pm^{-1}), \tag{48}$$
$$D_2/C_2=(\lambda_2+\alpha_\pm^{-1}), \tag{49}$$
$$\alpha_\pm=2[-(\lambda_1+\lambda_2)\pm\sqrt{(\lambda_1-\lambda_2)^2+4}]^{-1}. \tag{50}$$

We can consider two different pairs of amplitudes (48) and (49) depending on the choice of the sign of $\alpha_\pm$. In the slab geometry, Beltrami fields are given by

$$\mathbf{G}_j=(0,\sin\lambda_j x,\cos\lambda_j x)\quad(0\leqslant x\leqslant 1,\ j=1,2).$$

The necessary and sufficient condition for the stability of each separate Beltrami vortex is (see Sec. III)

$$|D_j/C_j|\leqslant 1\quad\text{or}\quad\lambda_j\leqslant\pi\quad(j=1,2).$$

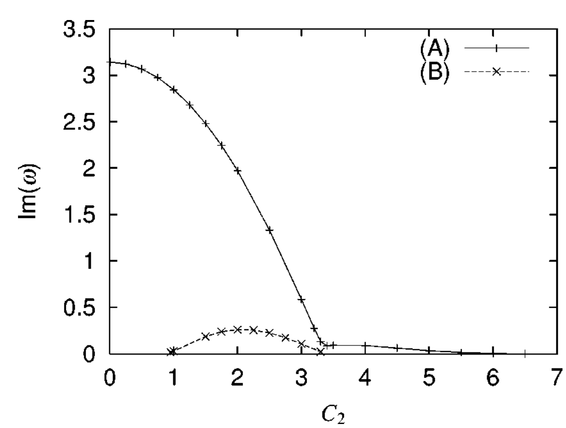

FIG. 6. The maximum growth rate versus C_2 with $\lambda_s=2\pi$, $\lambda_l=\pi$ and $C_1=1$. In Case (A), $\lambda_1=2\pi$ and $\lambda_2=\pi$. In Case (B), $\lambda_1=\pi$ and $\lambda_2=2\pi$.

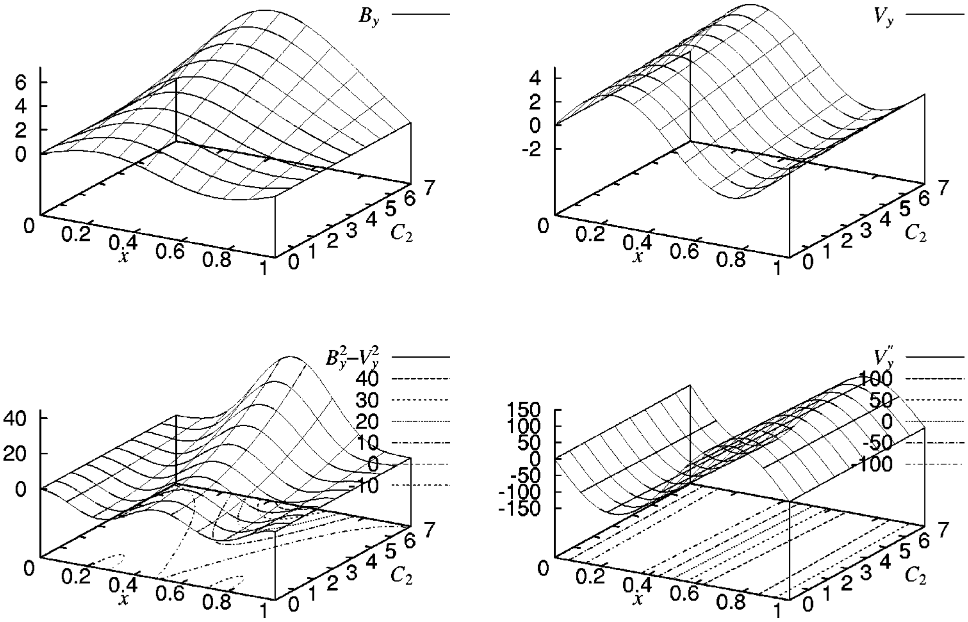

FIG. 7. The profiles of B_y, V_y, $B_y^2-V_y^2$ and V_y'' versus C_2 in Case (A) of Fig. 6.

Equations (48) and (49) show that the values D_j/C_j ($j=1,2$) are defined by λ_1 and λ_2, and their product must satisfy

$$(D_1/C_1)(D_2/C_2)=1, \tag{51}$$

which means that one vortex is sub-Alfvénic while the other is super-Alfvénic ($|D_1/C_1|=|D_2/C_2|=1$ occurs when $\lambda_1=\lambda_2$). A combination of two unstable vortices is not possible in a slab geometry. There are two types of combinations:

(A) super-Alfvénic vortex with smaller structure and sub-Alfvénic vortex with larger structure ($|D_1/C_1|>1$, $|D_2/C_2|<1$ and $\lambda_1>\lambda_2$),

(B) super-Alfvénic vortex with larger structure and sub-Alfvénic vortex with smaller structure ($|D_1/C_1|>1$, $|D_2/C_2|<1$ and $\lambda_1<\lambda_2$).

We distinguish the super- and sub-Alfvénic vortices by subscripts 1 and 2, respectively. In the double Beltrami fields, the profiles of B_y and V_y do not satisfy Eq. (32), and hence, the marginally stable eigenfunction cannot be found. In the case (A) with $\lambda_1=n\lambda_2$ (n is an integer number) or $\lambda_2\leqslant\pi$, the sufficient condition for the stability, $B_y^2\geqslant V_y^2$ in the whole region, holds for some appropriate choices of C_1 and C_2. Otherwise, however, the sufficient condition is satisfied only at the limit of $|C_2/C_1|\to\infty$.

Fixing $C_1=1$ and taking C_2 as a control parameter, we compare the growth rates of Cases (A) and (B) for the same pair of λ_1 and λ_2. We denote the larger (absolute value) one of $\lambda_{1,2}$ by λ_s (smaller size) and the smaller one by λ_l. For given λ_s and λ_l, the selection of α_+ or α_- in Eqs. (48) and (49) switches the Cases (A) and (B).

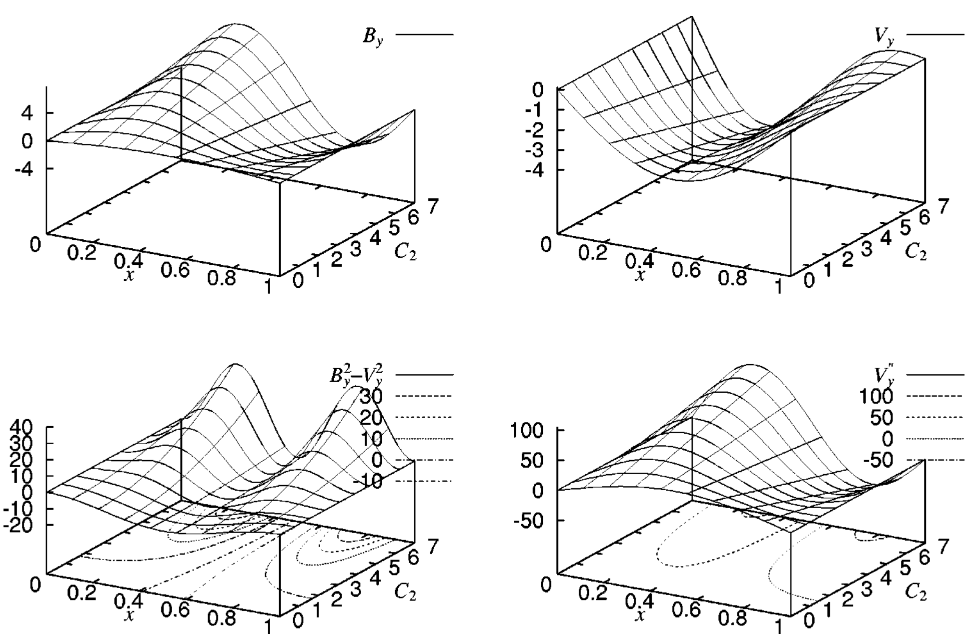

FIG. 8. The profiles of B_y, V_y, $B_y^2-V_y^2$ and V_y'' versus C_2 in Case (B) of Fig. 6.

Figure 6 shows the maximum growth rate as a function of C_2 (we choose $\lambda_s = 2\pi$ and $\lambda_l = \pi$). If we use α_+, the vortex of λ_s is unstable and that of λ_l is stable [Case (A)]. The profiles of B_y and V_y are shown in Fig. 7 as functions of C_2. When we increase C_2, the amplitude of the magnetic field increases. For $C_2 > 6.87$, the local Alfvén velocity exceeds the flow velocity everywhere in the domain $(0,1)$. In Fig. 7, we observe that the instability is suppressed for $C_2 \gtrsim 6.5$, implying that the criteria ($B_y^2 \geqslant V_y^2$ in the whole domain) is almost the necessary and sufficient condition.

If we use α_-, Case (B) occurs. Both vortices (λ_s and λ_l) are separately stable (the small vortex λ_s becomes sub-Alfvénic). The profiles of B_y and V_y are shown in Fig. 8. The combination of stable vortices causes instability. However, the growth rate is smaller than that of Case (A) (see Fig. 6). The flow of the combined vortices does not have an inflection point if $C_2 < 1.47$ (see Fig. 8). If there was no magnetic field, this flow is stable (Rayleigh's inflection-point theorem[11]). However, the double Beltrami field is unstable when $C_2 \gtrsim 1$ (see Fig. 6). This is the so-called joint instability[6,9]—an example of the destabilizing effect of a magnetic field.

V. SUMMARY

The stability of a plasma with ambient flow is a rather complex problem. Both flow shear and magnetic shear are double edge sword—they have both stabilizing and destabilizing effects. The stretching effect of a shear flow is believed to stabilize instabilities. However, the evolution of fluctuations in an ambient shear flow is rather complicated.[15,17–19] Nonexponential (possibly algebraic) behavior of fluctuations are left outside the scope of present paper. A sufficient condition of general stability (including even the nonlinear regime) will be discussed elsewhere. A shear flow contains a free energy to excite KH instabilities.

The existence of marginally stable eigenfunction has been shown analytically, which provides us with the necessary and sufficient condition for the exponential stability. We have seen that the magnetic shear effect reduces the growth rate of the instability, while the critical wave number is unchanged. We note that the assumption of a slab geometry omits the kink instabilities that are induced by the magnetic field curvature effect.[8] Instabilities in a cylindrical geometry will be discussed elsewhere. The Beltrami fields are simple in the sense of the relation (32) that allows us to construct the marginally stable solution.

In a general MHD flow, we can not find the marginally stable solution, so that the necessary and sufficient condition of stability is not known. The complexity of the stability problem is seen in the analysis of the double Beltrami fields. We have shown that the joint instability (destabilization by a magnetic field) may occur.

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[15] Due to the non-Hermitian property of the system, eigenfunctions of modes are not orthogonal with each other. Hence, KH-mode may couple with other modes (including continuous spectra). Such mode couplings may bring about rather complex time-asymptotic phenomena [see M. Hirota, T. Tatsuno, S. Kondoh, and Z. Yoshida, Phys. Plasmas **9**, 1177 (2002)]. However, those "secular" behavior is at most algebraic in t. If KH mode is unstable ($\mathrm{Im}\,\omega > 0$), it dominates the long-term behavior.
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