Transient phenomena and secularity of linear interchange instabilities with shear flows in homogeneous magnetic field plasmas

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(Received 14 August 2000; accepted 27 September 2000)

Transient and secular behaviors of interchange fluctuations are analyzed in an ambient shear flow by invoking Kelvin's method of shearing modes. Because of its non-Hermitian property, complex transient phenomena can occur in a shear flow system. The combined effect of shear flow mixing and Alfvén wave propagation overcomes the instability driving force at sufficiently large time, and damps all fluctuations of the magnetic flux. On the other hand, electrostatic perturbations can be destabilized for sufficiently strong interchange drive. The time asymptotic behavior in each case is algebraic (nonexponential). © *2001 American Institute of Physics.* [DOI: 10.1063/1.1336532]

I. INTRODUCTION

It is widely accepted that a shear flow yields stabilizing effects on various fluctuations through convective deformations of disturbances.[1,2] However, rigorous treatment of the shear flow effects encounters a fatal difficulty arising from the non-Hermitian (non-self-adjoint) properties of the problem. We may not consider well-defined ''modes'' and corresponding ''time constants.'' The standard normal mode approach breaks down, and the theory may fail to give correct predictions of evolution even if perturbation fields remain in the linear regime. The discrepancies between the theory and the experiment on the stability limit of neutral fluids are reviewed in Ref. 3. The aim of this work is to establish a solid foundation for the analysis of shear flow systems. We apply Kelvin's method of shearing modes.[4] This scheme, previously called the nonmodal approach, actually consists in the combination of two methods which have been widely used in solving wave equations; the modal and the characteristics methods.

Much work has been done on instability problems with shear flows by means of the modal approach. It is implicitly assumed in the application of the modal scheme that the motion can be decomposed into a set of independent normal modes with certain time constants.[5] As is well known, this method is effective in solving problems involving Hermitian operators, however, when applying it to non-Hermitian systems, we may overlook the secular and transient behaviors. On the other hand, the characteristics method has been used in the context of rapid distortion theory for analyzing the fluid turbulence[6] and in the eikonal representation of the ballooning mode stability.[7] If we can treat the non-Hermitian part of the whole operator as a singular perturbation to a Hermitian operator,[8,9] we may be able to construct the theory in the framework of the perturbation theory for the operator.[10] But, unfortunately, the convergence of the perturbative series seems to be very ambiguous in case of the shear flows due to the secularity of their time evolutions. Thus, a thorough mathematical treatment of the non-Hermitian properties of shear flow systems has not been accomplished so far. In this paper, we have analyzed the shear flow effect on interchange instabilities and its non-Hermitian mathematical background, deriving the time asymptotic behavior by means of Kelvin's method.

Recently, Kelvin's method has been applied to a variety of linear shear flow problems.[11–17] For neutral and magnetized fluids, many new fascinating phenomena were discovered; exchanges of energy between background flows and perturbation fields,[13] shear flow induced coupling between sound waves and internal waves and the excitation of beat wave,[14] the asymptotic persistence due to the periodic energy transfer for two-dimensional shear flows,[15] and the emission of magnetosonic waves by the stationary vortex perturbations.[16] These results show that the modes, which are independent for static fluids, are no longer independent and the coupling of these modes induces the observed energy transfer in the presence of the shear flow. The authors have also applied this method to investigate the basic properties of kink-type instabilities in the presence of a background shear flow and obtained the result that the shear flow mixing always overcomes the kink driving at sufficiently large time.[17]

In this paper, we will first revisit Kelvin's method from the viewpoint of the characteristics method (Sec. II). We will review the spectral theory focusing on the general mathematical concept of eigenmode in order to gain a better understanding of Kelvin's method. In Sec. III, we will formulate the equations for the interchange instabilities. In Sec. IV, we will derive the ordinary differential equation (ODE) in time for the evolution of the amplitude of the interchange instabilities by applying the analysis of shearing modes. In Sec. V, by drawing the analogy with Newton's equation it will be shown that the solution to the above-mentioned ODE for the flux function exhibits an asymptotic damped behavior without any threshold of instability drive. We will also consider the electrostatic perturbations in Sec. VI. The solution of the derived ODE for the stream function shows the asymptotic growth or decay of algebraic type depending on the magnitude of instability drive. We will summarize the result in Sec. VII. Moreover in the Appendix, we will also show the difficulties encountered by including the magnetic shear in the present formulation.

II. NON-HERMITIAN PROPERTY OF SHEAR FLOW SYSTEMS

Before formulating the interchange instability equations, let us describe a rough sketch of the problem and explain the mathematical tool to analyze the non-Hermitian dynamics. As is well known, the force operator governing the linear dynamics of static magnetohydrodynamic (MHD) plasmas in Hermitian,[18] and therefore the perturbation fields can be decomposed into a set of orthogonal eigenmodes which show purely exponential (unstable) or purely oscillating (stable) evolutions. A nontriviality stems from the Alfvénic and acoustic continuous spectra; the phase mixing damping occurs. This behavior, however, is totally within the framework of the well-known theory of Hermitian operators.[19]

In the case where ambient shear flow exists, however, the operator becomes non-Hermitian and resolution in terms of orthogonal eigenmodes fails. From a dynamical point of view, the system experiences evolutions of a complex type. In the following sections, we will show examples of such kind of ''non-Hermitian'' dynamics where transient phenomena and secular evolutions play a dominant role. Similar evolutions are found in the case of non-Hermitian operators in finite dimensional vector spaces.[20] It has to be stressed how the application of the traditional modal paradigm to non-Hermitian systems, which assumes exponential evolution of the perturbation fields, hinders the possibility of catching these rich variety of transient and algebraic phenomena. In this section, we will discuss Kelvin's method and show its suitability to the analysis of shear flow non-Hermitian systems. We will revisit it from the viewpoint of the characteristics method showing that it represents a generalization of the modal approach.

Unlike matrices, clear classification of differential operators becomes extremely difficult due to their infinite dimensionality of the spaces they act on, and also to their unboundedness. The linearized dynamics of fluid systems in the presence of sheared flow is governed by a general equation of the following type:

$$\partial_t u + \mathbf{v}\cdot\nabla u = \mathcal{A}u, \tag{1}$$

where $\mathcal{A}$ denotes a Hermitian differential operator (time-independent) defined in a Hilbert space V, $\mathbf{v}$ is the stationary mean flow, and $u(\in V)$ denotes a perturbation field.

It is the convective derivative, $\mathbf{v}\cdot\nabla$, that introduces the non-Hermitian property into problem (1) and prevents the possibility of representing the dynamics of the systems in terms of the orthogonal and complete set of eigenfunctions. This is a well-known difficulty in the stability analysis of neutral fluids, such as Couette or Poiseille flows, where the predictions obtained by means of the modal methods do not match the experiments.[3]

In the case of a spatially inhomogeneous stationary flow $\mathbf{v}$, Eq. (1) becomes non-Hermitian and a straightforward spectral resolution is not effective. However, Kelvin's method permits to resolve, for some classes of mean flows, the evolution of the system (1) into new types of modes by means of which both transient and secular asymptotic behaviors are effectively described. Let us now explain the mathematical foundations of this scheme.

As mentioned in Sec. II, Kelvin's method consists in the combined application of two methods which have been extensively used in the analysis of wave equations. Precisely the ''Lagrangian'' part of Eq. (1), $\partial_t + \mathbf{v}\cdot\nabla$, is solved by means of the characteristics method and the ''Hermitian'' part $\mathcal{A}$ by means of the standard spectral resolution.

The characteristics method is applied to solve the characteristic ODE associated to the Lagrangian derivative moving along the characteristic curve of the ambient motion, which is given by

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}, \quad \mathbf{x}(0) = \boldsymbol{\xi}. \tag{2}$$

By inverting the modes, which are expressed in Lagrangian coordinates as $\varphi(\mathbf{k},\boldsymbol{\xi})$, they will be represented in Eulerian coordinates as

$$\tilde{\varphi}(t;\mathbf{k},\mathbf{x}) = \varphi(\mathbf{k},\boldsymbol{\xi}(t;\mathbf{x})), \tag{3}$$

where $\boldsymbol{\xi}(t;\mathbf{x})$ denotes the inverse of $\mathbf{x}(t;\boldsymbol{\xi})$. The existence of the inverse mapping $\mathbf{x}(t)\mapsto\boldsymbol{\xi}$ is guaranteed in the case of incompressible mean flows. Due to Eq. (3), $\tilde{\varphi}(t;\mathbf{k},\mathbf{x})$ satisfies the characteristic equation

$$\partial_t \tilde{\varphi}(t;\mathbf{k},\mathbf{x}) + \mathbf{v}\cdot\nabla\tilde{\varphi}(t;\mathbf{k},\mathbf{x}) = 0. \tag{4}$$

The essential condition for the applicability of Kelvin's method consists in the constraint for the function $\tilde{\varphi}(t;\mathbf{k},\mathbf{x})$ to form the complete set of eigenfunctions of the operator $\mathcal{A}$. If such a set of eigenfunctions exists, we can decompose the perturbation field u by means of

$$u = \int \hat{u}_k(t)\tilde{\varphi}(t;\mathbf{k},\mathbf{x})\,d\mathbf{k}. \tag{5}$$

We notice that due to Eq. (3) the eigenvalues of $\mathcal{A}$ become time dependent. The new eigenvalue problem for $\mathcal{A}$ reads

$$\mathcal{A}\tilde{\varphi}(t;\mathbf{k},\mathbf{x}) = \lambda_k(t)\tilde{\varphi}(t;\mathbf{k},\mathbf{x}). \tag{6}$$

Plugging Eq. (5) into Eq. (1) and exploiting Eqs. (4) and (6), we have

$$\int [\partial_t \hat{u}_k(t)]\tilde{\varphi}(t;\mathbf{k},\mathbf{x})\,d\mathbf{k} = \int \hat{u}_k(t)\lambda_k(t)\tilde{\varphi}(t;\mathbf{k},\mathbf{x})\,d\mathbf{k}. \tag{7}$$

Due to the orthogonality of the modes $\tilde{\varphi}(t;\mathbf{k},\mathbf{x})$, the evolution of $\hat{u}_k$ is governed by the equation

$$\frac{d}{dt}\hat{u}_k(t) = \lambda_k(t)\hat{u}_k(t). \tag{8}$$

If $\tilde{\varphi}(t;\mathbf{k},\mathbf{x})$ do not satisfy both conditions given by characteristic equation (4) and eigenequation (6), Eq. (7) will have additional terms which represent the complicated mode coupling and thus the applicability of Kelvin's method is compromised.

Due to the time dependence present in the eigenvalues $\lambda_k(t)$, the evolution of $\hat{u}_k(t)$ will not exhibit a simple exponential dependence as in the Hermitian case, but more complicated behaviors, which are characteristic of non-Hermitian systems. By analyzing this ODE, the motion of each mode can be classified, and the time asymptotic behavior can also be shown. The following sections will be devoted to the derivation of ODE (8) and the discussion of the behavior of its solution for interchange instabilities in plasmas with shear flow.

III. FORMULATION OF INTERCHANGE INSTABILITIES

Interchange instabilities have been analyzed for static plasmas by many authors.[21–23] In the case of static (stationary ambient flow $\mathbf{v}_0=0$) plasmas, the ideal MHD equations can be reduced into a simple partial differential equation of the form[18]

$$\partial_t^2 \boldsymbol{\xi} = \mathcal{F}\boldsymbol{\xi}, \tag{9}$$

where $\boldsymbol{\xi}$ is the displacement vector and $\mathcal{F}$ is the force operator which is Hermitian (self-adjoint) when the plasma is surrounded by an ideal conducting wall. In order to analyze the stability of the system, we can apply the spectral method and represent the dynamics in terms of a superposition of harmonic oscillations of modes. Another method of analyzing the stability of the static plasmas is to apply the energy principle[21] which is a variational approach based on the Hermitian property of the force operator $\mathcal{F}$. These methods show that the interchange modes have spatially localized structures near the marginal stability[22] except when $p' \simeq 0$ on the rational surface.[23]

It is remarkably difficult to estimate the exact linear stability of the system in the presence of a stationary shear flow, since, as seen in the preceding sections, the dynamics become non-Hermitian and both the spectral and the variational methods lose their mathematical foundations. Dispersion relations have been studied in many publications,[1,5,8] however, as discussed in Sec. I, the evolution of a non-Hermitian system cannot be reconstructed from the formal dispersion relation, because we do not have a spectral theory. Since the proper asymptotic behavior of interchange instabilities are not clearly shown yet, we will first analyze simplified systems focusing on the non-Hermitian property of the system. In this section, we will derive the equations for stationary flowing plasmas. Specifically we will investigate the effect of shear flows on interchange instabilities of plasma under the influence of homogeneous magnetic field.

In the presence of gravitational force, the ideal MHD equations read as

$$\rho \frac{d\mathbf{v}}{dt} = \mathbf{j}\times\mathbf{B} - \nabla p + \rho\mathbf{g}, \tag{10}$$

$$\frac{d\rho}{dt} + \rho \nabla\cdot\mathbf{v} = 0, \tag{11}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla\times(\mathbf{v}\times\mathbf{B}) = 0, \tag{12}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{13}$$

where ρ, $\mathbf{B}$, and $\mathbf{g}$ are the density, magnetic field, and gravitational constant vector, and $d/dt = \partial_t + \mathbf{v}\cdot\nabla$ denotes the Lagrangian derivative. Here we assume the incompressibility of the velocity field $\mathbf{v}$, instead of using the equation of state.

The ambient fields (denoted by the subscript 0) must satisfy

$$\rho_0 \mathbf{v}_0 \cdot \nabla \mathbf{v}_0 = \mathbf{j}_0 \times \mathbf{B}_0 - \nabla p_0 + \rho_0 \mathbf{g}. \tag{14}$$

If we consider a parallel stationary shear flow of the form $\mathbf{v}_0 = (0, v_{0y}(x), 0)$, straight homogeneous magnetic field $\mathbf{B}_0 = (0, B_y, B_z)$, and gravitational force acting in the positive x direction, the convective derivative gives no contribution to the stationary state and Eq. (14) is reduced to

$$\nabla p_0 = \rho_0 \mathbf{g}. \tag{15}$$

The above equation denotes that the pressure gradient is balanced by the gravitational force in the x direction. This is the same condition which holds for static neutral fluids.

The perturbed magnetic and velocity fields are assumed to be two dimensional in the $x-y$ plane, and thus we can introduce the poloidal flux function and stream function,

$$\mathbf{B}_{1\perp} = \nabla\psi\times\mathbf{e}_z, \quad \mathbf{v}_{1\perp} = \nabla\phi\times\mathbf{e}_z, \tag{16}$$

where the subscript 1 denotes the perturbed quantities, $\perp$ expresses the direction perpendicular to the dominant magnetic field directed along the z axis, and $\mathbf{e}_z$ denotes the unit vector in the z direction. Using these representations, we can eliminate the pressure from governing equations.

Taking the curl of the equation of motion and projecting it along $\mathbf{e}_z$, we obtain

$$\mu_0\rho_0[(\partial_t + v_{0y}\partial_y)\nabla_\perp^2\phi - v_{0y}''\partial_y\phi] = \mathbf{B}_0\cdot\nabla(\nabla_\perp^2\psi) + \mu_0\partial_y\rho_1 g, \tag{17}$$

where $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. In deriving Eq. (17) we have used the Boussinesq approximation which consists in the neglect of the spatial variation of the stationary state density in the inertial term of the equation of motion, but not in the continuity equation since it is the driving term for the interchange instability. Physically it is valid provided that the variability in the density is due to variations in the temperature of only moderate amounts.[24] The component of the flow perpendicular to the ambient magnetic field can be considered consistently coming from the $\mathbf{E}\times\mathbf{B}$ drift, taking into account the ideal Ohm's law. It is noted that, if we neglect the effect of the magnetic field, we recover the Rayleigh's equation for Kelvin–Helmholtz instability.[25]

The density fluctuation can be expressed as

$$(\partial_t + v_{0y}\partial_y)\rho_1 = -\rho_0'\partial_y\phi, \tag{18}$$

where the prime denotes the derivative with respect to x. Now ρ_0' is considered as a constant which introduces a destabilizing force. The induction equation is the same as in the ordinary reduced MHD equations[26] and under the above assumptions on the stationary fields reads as

$$(\partial_t + v_{0y}\partial_y)\psi = \mathbf{B}_0\cdot\nabla\phi. \tag{19}$$

Equations (17)–(19) constitute a closed system of equations. We can see that the static system ($v_{0y}=0$) governed by these equations shows Hermitian property, and the convective derivative ($v_{0y}\neq 0$) brings the non-Hermitian property into our system. We will investigate the effect of the shear flow on the interchange instabilities in the following sections.

IV. DERIVATION OF ORDINARY DIFFERENTIAL EQUATION

In this section, we derive the ODE for the amplitude of Kelvin's modes, given in Eq. (8), in the case of interchange instabilities of plasmas. Let us first consider the electromagnetic case where $\mathbf{B}_0\cdot\nabla\neq 0$. From Eqs. (18) and (19), we have

$$\phi=-\partial_y^{-1}\rho_0'^{-1}(\partial_t+v_{0y}\partial_y)\rho_1=(\mathbf{B}_0\cdot\nabla)^{-1}(\partial_t+v_{0y}\partial_y)\psi. \tag{20}$$

Since we have assumed the mean velocity $v_{0y}=v_{0y}(x)$ and the homogeneous ambient field $\mathbf{B}_0=(0,B_y,B_z)$, the operator $\partial_t+v_{0y}\partial_y$ commutes with both ∂_y^{-1} and $(\mathbf{B}_0\cdot\nabla)^{-1}$. Thus acting on both sides of Eq. (20) with the operator $(\partial_t+v_{0y}\partial_y)^{-1}$ gives

$$\rho_1=-\rho_0'\partial_y(\mathbf{B}_0\cdot\nabla)^{-1}\psi. \tag{21}$$

From Eq. (19),

$$\nabla_\perp^2\phi=\nabla_\perp^2(\mathbf{B}_0\cdot\nabla)^{-1}(\partial_t+v_{0y}\partial_y)\psi. \tag{22}$$

Substituting Eqs. (20) and (22) into Eq. (17), and acting with $\mathbf{B}_0\cdot\nabla$ on both sides, we obtain

$$(\partial_t+v_{0y}\partial_y)\nabla_\perp^2(\partial_t+v_{0y}\partial_y)\psi=\frac{(\mathbf{B}_0\cdot\nabla)^2}{\mu_0\rho_0}\nabla_\perp^2\psi-\frac{\rho_0'g}{\rho_0}\partial_y^2\psi. \tag{23}$$

Since the operator on the right-hand side is Hermitian, we can decompose the flux function ψ by means of the shearing eigenmodes

$$\psi(\mathbf{x},t)=\int\hat{\psi}_k(t)\tilde{\varphi}(t;\mathbf{k},\mathbf{x})d\mathbf{k}, \tag{24}$$

where each eigenmode can be expressed by the sinusoidal function in our simplified case

$$\tilde{\varphi}(t;\mathbf{k},\mathbf{x})=\exp[ik_xx+ik_y(y-v_{0y}t)+ik_zz]=\exp[i\tilde{k}_x(t)x+ik_yy+ik_zz]. \tag{25}$$

Here the mean flow is assumed to be $v_{0y}(x)=\sigma x$ and $\tilde{k}_x(t)=k_x-k_y\sigma t$. It is explicitly shown that the wave number in the flow shear direction is linearly increasing with time due to the distorting effect of the mean flow. Since continuous variation of $\tilde{k}_x(t)$ prevents from imposing the boundary condition in the bounded domain, we will concentrate on the analysis of localized perturbations by considering the infinite domain. Note that $\tilde{\varphi}$ are the eigenfunctions of the right-hand side of Eq. (23), and also satisfy the characteristic equation (4). It should be noted that the presence of the Laplacian operator on the left-hand side of Eq. (23) does not hinder the application of Kelvin's method since the modes $\tilde{\varphi}$ are as well eigenfunctions of the Laplacian $\nabla_\perp^2$.

Thus, the time evolution equation for the amplitude $\hat{\psi}_k$ can be written as

$$\frac{d}{dt}\left[(\tilde{k}_x(t)^2+k_y^2)\frac{d\hat{\psi}}{dt}\right]=-\frac{F^2}{\mu_0\rho_0}(\tilde{k}_x(t)^2+k_y^2)\hat{\psi}-k_y^2\frac{\rho_0'g}{\rho_0}\hat{\psi}, \tag{26}$$

where $F=\mathbf{k}\cdot\mathbf{B}_0=k_yB_{0y}+k_zB_{0z}$, and we have dropped the subscript k for simplicity. We notice that in the absence of shear flow ($\sigma=0$) the usual interchange instability equation for static equilibrium can be obtained.

Our procedure can be readily shown to coincide with the traditional formulation of Kelvin's method consisting in the coordinate transform $(t,x,y,z)\mapsto(T,\xi,\eta,\zeta)$ defined by

$$T=t,\quad \xi=x,\quad \eta=y-\sigma tx,\quad \zeta=z, \tag{27}$$

and the Fourier transform with respect to the new coordinates

$$\hat{u}(k_\xi,k_\eta,k_\zeta;T)=\int\int\int_{-\infty}^{+\infty}u(\xi,\eta,\zeta;T)e^{i(k_\xi\xi+k_\eta\eta+k_\zeta\zeta)}d\xi\,d\eta\,d\zeta. \tag{28}$$

Normalizing the time t by the poloidal Alfvén time $\tau_A=a\sqrt{\mu_0\rho_0}/F$, we can rewrite Eq. (26) in dimensionless form as

$$\frac{d}{dt}\left[(\tilde{k}_x(t)^2+k_y^2)\frac{d\hat{\psi}}{dt}\right]=-(\tilde{k}_x(t)^2+k_y^2)\hat{\psi}+k_y^2\frac{\tau_A^2}{\tau_G^2}\hat{\psi}, \tag{29}$$

where the wave vectors are normalized by the characteristic length scale a and $\tau_G^2=-\rho_0/\rho_0'g$. Further we can rewrite Eq. (29) in the form

$$\frac{d^2\hat{\psi}}{dt^2}+\mu(t)\frac{d\hat{\psi}}{dt}+[1-S(t)]\hat{\psi}=0, \tag{30}$$

where

$$\mu(t)=-\frac{2\sigma k_y\tilde{k}_x(t)}{\tilde{k}_x(t)^2+k_y^2},\quad S(t)=\frac{k_y^2G}{\tilde{k}_x(t)^2+k_y^2},$$

and $G=\tau_A^2/\tau_G^2$. Drawing an analogy with Newton's equation, $\mu(t)$ represents the frictional term and $S(t)$ the interchange drive term. Equation (30) is the correspondent of Eq. (8). As we have mentioned in Sec. II, the time evolution for the amplitude of each eigenmode is no longer a simple exponential function. The behavior of $\hat{\psi}$ will be discussed in the following sections.

V. ASYMPTOTIC AND TRANSIENT BEHAVIOR OF EACH MODE

In the absence of a density gradient or shear flow, $\mu(t)=S(t)=0$ in Eq. (30) and we have a pure oscillation representing the Alfvén wave. If we include the density gradient, then $S(t)\neq 0$ and we obtain the interchange instability for negative ρ_0'. Since a homogeneous magnetic field is assumed in this paper, we have no stabilizing effect of the magnetic shear. The operator is Hermitian in these two cases, therefore we have the simple exponential evolution with time constants for each mode.

When we include the shear flow, we have $\mu(t)\neq 0$ and we can draw an analogy with the dynamics of a damped oscillator with time dependent frictional coefficient $\mu(t)$. When time goes, $\mu(t)$ becomes always positive, which means a formal dissipation, and therefore the oscillation energy of the Alfvén wave $[(d\hat{\psi}/dt)^2+\hat{\psi}^2]/2$ decreases monotonically. In the following sections we will describe both the asymptotic and transient behaviors of the amplitudes $\hat{\psi}$.

A. Asymptotic behavior

In order to study the time asymptotic behavior, we assume $t \gg k_x/\sigma k_y, 1/\sigma$. In this time asymptotic limit we obtain the following ODE:

$$\frac{d^2}{dt^2}\hat{\psi}+\frac{2}{t}\frac{d}{dt}\hat{\psi}+\left(1-\frac{G/\sigma^2}{t^2}\right)\hat{\psi}=0, \tag{31}$$

where $G=\tau_A^2/\tau_G^2$ denotes the magnitude of the instability drive term. In the absence of the instability drive G, the time asymptotic behavior of the solution of Eq. (31) is expressed as

$$\hat{\psi}\sim\frac{1}{t}\sin t, \tag{32}$$

which coincides with the result of Koppel[27] which considered a time dependent nonperturbative state. Since Eq. (31) is the spherical Bessel equation, its general solution for $G \neq 0$ is expressed by

$$\hat{\psi}=\frac{1}{\sqrt{t}}(C_1 J_\nu(t)+C_2 Y_\nu(t)), \tag{33}$$

where J_ν and Y_ν denote the Bessel functions, and $\nu=(G/\sigma^2+1/4)^{1/2}$. Therefore the time asymptotic behavior of the mode is expressed generally as

$$\hat{\psi}\sim\frac{1}{t}\sin\left(t-\frac{\pi\nu}{2}+\delta\right), \tag{34}$$

where δ denotes a constant phase depending on the initial values. Therefore the mode oscillates with amplitude $\hat{\psi}$ decaying with the inverse power of time. While the x component of the perturbation magnetic field $\hat{b}_x$ is proportional to ψ, the y component $\hat{b}_y$ is proportional to $\tilde{k}_x(t)\hat{\psi}$. Thus $\hat{b}_y$ tends to the pure oscillatory behavior

$$\hat{b}_y\sim\sin\left(t-\frac{\pi\nu}{2}+\delta\right), \tag{35}$$

as $\tilde{k}_x(t)$ increases with proportion to time (see Fig. 1). It should be noted that there is no threshold value for the stabilization of the interchange instability, since we obtain the same spherical Bessel equation (31) for all modes. All modes evolve as in Eq. (31) independently of the values of wave numbers **k**.

The final amplitude of each mode depends sensitively on the parameters. As the shear parameter increases, the final amplitude of $\hat{b}_y$ tends to be larger as is also shown by Chagelishvili *et al.*,[12] while the mixing damping effect on $\hat{b}_x$ increases. The numerical integration of Eq. (30) is shown in Fig. 1. It should be noted that the instability drive G asymptotically has the only effect to shift the phase of the oscillations as can be seen in Eqs. (34) and (35), and it does not affect the principal time dependence. The combined effect of the Alfvén wave propagation and shear flow mixing always overcomes the interchange drive and the oscillations of the magnetic flux asymptotically decay with proportion to the inverse power of time.

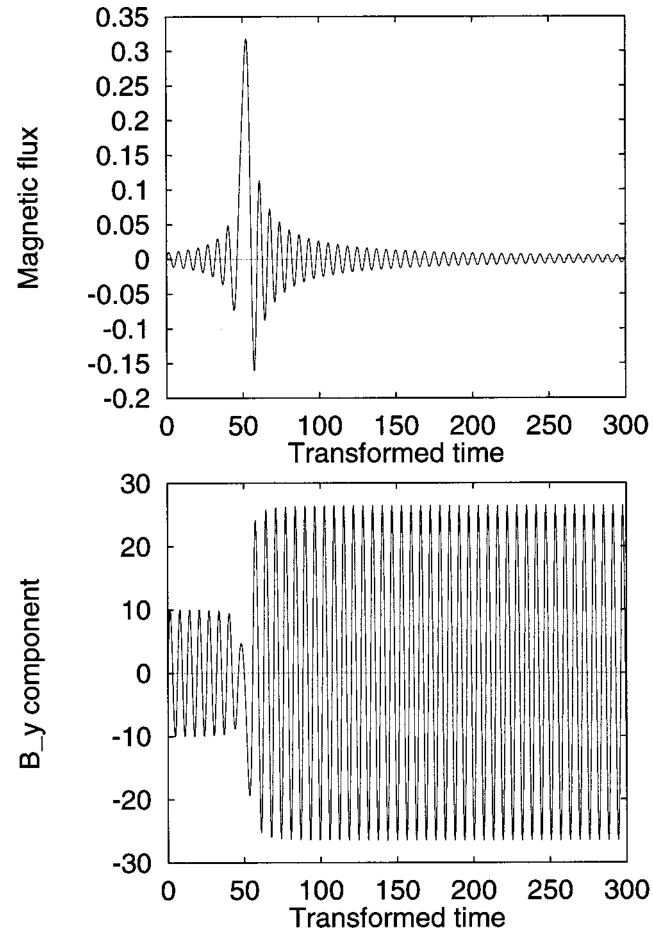

FIG. 1. Direct numerical integration of Eq. (30) for each mode. The parameters are as follows: $k_x=10$, $k_y=1$, $k_z=0$, $\sigma=0.2$, $G=1$, and initial perturbations $\hat{\psi}=0$ and $d\hat{\psi}/dt=1.0$ at $t=0$.

B. Transient behavior

In this section, we analyze the transient behavior of each mode. Since an analytic expression is not available, we discuss the transients by qualitatively analyzing the ODE (30). In the absence of the instability drive, we have

$$\frac{d}{dt}\left[\left(\frac{d\hat{\psi}}{dt}\right)^2+\hat{\psi}^2\right]=-\mu(t)\left(\frac{d\hat{\psi}}{dt}\right)^2, \tag{36}$$

where

$$\mu(t)=-\frac{2\sigma k_y\tilde{k}_x(t)}{\tilde{k}_x(t)^2+k_y^2},$$

$$\tilde{k}_x(t)=k_x-\sigma k_y t.$$

Therefore, the frictional coefficient $\mu(t)$ acts as a damping force for $\mu>0$. Since the sign of the denominator in $\mu(t)$ is always positive, the behavior will be determined by that of the numerator. The numerator can be expressed as $2\sigma^2 k_y^2 t - 2\sigma k_y k_x$ and according to its initial value we can single out two classes of the transients.

When the product $\sigma k_y k_x$ is negative, the frictional coefficient $\mu(t)$ is always positive from the beginning, therefore the shear flow acts as a damping force at any time and the mode shows simple damped behavior. On the other hand, if the product $\sigma k_y k_x$ is positive, the frictional coefficient $\mu(t)$ is initially negative and changes its sign at the instant $t_* = k_x/\sigma k_y$. Therefore the mode experiences an initial amplification lasting until the time t_*, which is even faster than it would be in the presence of the only interchange drive. This transient behavior can also be seen in Fig. 1, where the initial amplification lasts until the turning point $t_*=50$ followed by the asymptotic decaying phase.

We have observed by numerical integration that the amplitude can be amplified to values of 10^{30} times larger than the initial one. From a physical point of view, such huge amplifications may break down the linearity of the perturbations and may lead to a nonlinear stage. This case is beyond the scope of the linear theory and no sure conclusion can be drawn from Kelvin's method. Such huge amplifications are experienced by modes with large t_* and G.

VI. ELECTROSTATIC INSTABILITY

When the wave vector is perpendicular to the ambient magnetic field, the formulation for the flux function (23) fails. For this ''electrostatic limit,'' we discuss the evolution of the stream function ϕ. The governing equations are Eqs. (17) and (18), since the flux freezing equation can be decoupled due to the fact that $\mathbf{B}_0 \cdot \nabla = 0$. In the case of electrostatic perturbations, drift wave may be destabilized, however, we have dropped the drift wave branch here in order to concentrate our attention on the single fluid MHD model. Applying $\partial_t + v_{0y}\partial_y$ to both sides of Eq. (17) and substituting it into Eq. (18), we have

$$(\partial_t + v_{0y}\partial_y)^2 \nabla_\perp^2 \phi = -\frac{\rho_0' g}{\rho_0}\partial_y^2 \phi, \tag{37}$$

for a linear shear flow. We represent ϕ in terms of the shearing modes given in Eq. (25),

$$\phi(\mathbf{x},t) = \int \hat{\phi}_k(t)\hat{\varphi}(t;\mathbf{k},\mathbf{x})d\mathbf{k}. \tag{38}$$

By substituting Eq. (38) into Eq. (37), the following ODE is obtained:

$$\frac{d^2}{dt^2}[(\tilde{k}_x(t)^2 + k_y^2)\hat{\phi}] = k_y^2 \gamma_G^2 \hat{\phi}, \tag{39}$$

where $\gamma_G^2 = -\rho_0' g/\rho_0 (=\tau_G^{-2})$ denotes the characteristic growth rate of the interchange instability. Here again we have dropped the subscript k for the sake of simplicity. In order to investigate the time asymptotic behavior of each mode, we assume $t \gg k_x/k_y\sigma$ and $t \gg 1/\sigma$, then Eq. (39) becomes

$$\frac{d^2}{dt^2}\hat{\phi} + \frac{4}{t}\frac{d}{dt}\hat{\phi} + \frac{2-\alpha}{t^2}\hat{\phi} = 0, \tag{40}$$

where $\alpha = \gamma_G^2/\sigma^2$ denotes the ratio of the strength of the interchange destabilizing effect and flow shear stabilizing one. Note that this ODE is not dependent on the wave numbers $\mathbf{k}$. The solution of Eq. (40) reads

$$\hat{\phi} = C_1 t^{m_+} + C_2 t^{m_-}, \tag{41}$$

where

$$m_\pm = \frac{-3 \pm \sqrt{1+4\alpha}}{2}. \tag{42}$$

The time asymptotic behavior is therefore determined by the larger index m_+. Thus we can state the condition for the boundedness of $\hat{\phi}$ as

$$\alpha \leqslant 2 \Rightarrow -\frac{1}{2}\frac{\rho_0' g}{\rho_0} \leqslant \sigma^2. \tag{43}$$

The condition for the boundedness of $\hat{\phi}$ has been improved with respect to the static case ($\rho_0' \geqslant 0$) due to the mixing effect of the shear flow. The electrostatic perturbation can be linearly unstable while the electromagnetic one is completely stabilized. The direct numerical integration of the ODE (39) is illustrated in Fig. 2. The transient behavior can be observed until the time $t_*=5$, and the asymptotic behavior follows. The asymptotic behavior is shown to be algebraic with the power m_+ as pointed out by means of the analytic treatment.

We notice that the stability condition is not well defined here. In fact by imposing the boundedness of $\hat{v}_y = i\tilde{k}_x(t)\hat{\phi} \sim t^{1+m_+}$, the same condition $\rho_0' \geqslant 0$ as the static case is obtained. If we consider the other fields which are represented by higher derivatives, e.g., the vortex perturbations, more strict conditions for their boundedness will be recovered. Since the mixing effect of the shear flow distorts the structure of the perturbations into smaller scales, the fields characterized by the higher derivatives will have stronger secularities. Unlike the static case where the evolution of the fields can be expressed in a common exponential form, they exhibit different evolutions with respect to each other in shear flow systems. This effect could be a pathological problem of describing the stability condition for shear flow systems.

VII. SUMMARY

Kelvin's method of shearing modes is interpreted as a combination of modal and characteristic methods for the analysis of a non-Hermitian system. A shear flow distorts each Fourier mode, resulting in a change of the wave number, which represents the stretching effect of the shear flow.

By means of this method, we have first analyzed the incompressible electromagnetic perturbations in the presence of an interchange drive and obtained the ordinary differential equation (30) for the amplitude of the modes $\hat{\psi}_k$. All modes show asymptotic decay proportional to the inverse power of time (nonexponential) without any threshold value. This

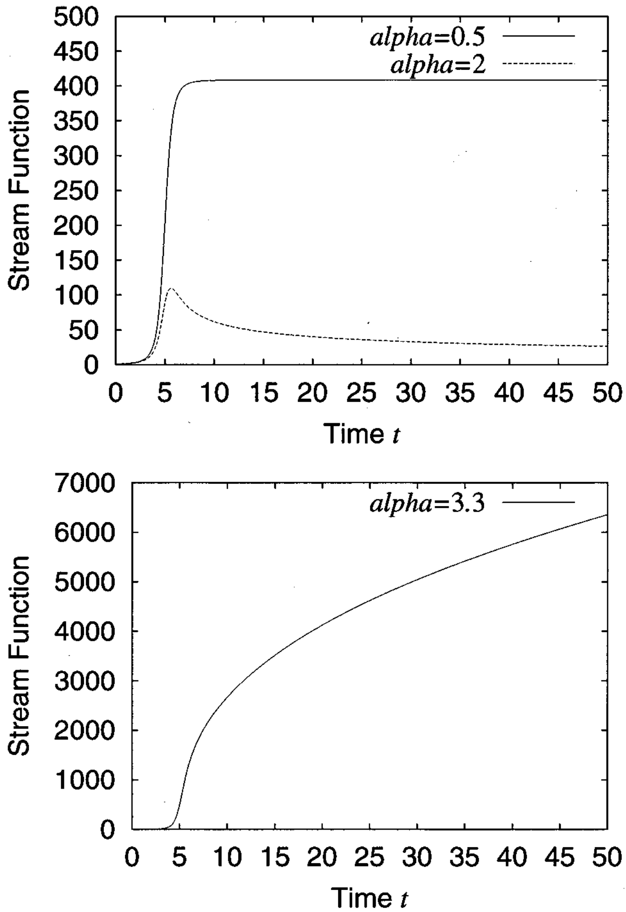

FIG. 2. Direct numerical integration of Eq. (39) for different α. The parameters are as follows: $k_x=10$, $k_y=2$, $\sigma=1$ and initial perturbations $\hat{\phi}=0$ and $d\hat{\phi}/dt=1.0$ at $t=0$. The amplitude of the stream function in case of $\alpha=3.3$ shows the algebraic growth corresponding to $m_+\simeq 0.35$.

means that the interchange instabilities are always damped away at sufficiently large time due to the combined effect of the Alfvén wave propagation and distortion of modes by means of the background shear flow; i.e., phase mixing effect. However, the transient behavior is not common for all modes. Depending on the initial wave number, some of them show transient amplifications which are even faster than they would be in the presence of the only interchange drive. These amplifications are so conspicuous that they may lead to the break down of the linearity of the perturbation fields.

It should be noted that, since our treatment considers the case of parallel linear shear flow, Kelvin–Helmholtz instabilities, which originate from the second-order spatial derivative of the background shear flow,[24,25] are beyond the scope of the present theory. From a mathematical point of view, we stress that the Kelvin–Helmholtz instability is a problem involving purely non-Hermitian operators in the sense that the operator $\mathcal{A}$ of Eq. (1) itself becomes non-Hermitian and therefore the method developed in Sec. II cannot be applied. This is a well-known instability in fluid dynamics whose rigorous mathematical treatment presents highly nontrivial difficulties.

We note that the ODE which gives the evolution of the amplitudes of the interchange modes (30) and that of kink-type modes [Eq. (32) in Ref. 17] are mathematically equivalent. Of course these two modes may have spatially different structures, at least this is the case for static equilibria. But this fact means that they have no difference in time evolution, and we can say that these terms have the same effect in the sense that they enlarge the spectrum to unstable eigenvalues. This equivalence stems from the assumption of a spatially homogeneous magnetic field. The possibility of including the magnetic field inhomogeneity is investigated in the Appendix.

We have also investigated the time evolution for electrostatic ($\mathbf{k}\cdot\mathbf{B}_0=0$) perturbations, which do not excite the Alfvén wave, since they do not bend the magnetic field line during their growth. The flow shear has been shown to have a stabilizing effect also on electrostatic disturbances, however, the phase mixing effect alone cannot completely stabilize the interchange instabilities. The condition for the boundedness of the mode amplitudes $\hat{\phi}_k$ can be expressed in Eq. (43) by means of a ratio of instability strength to shear parameter of the mean flow. We have shown that the time evolution of these unstable modes is again of algebraic type. Notice that the conditions for the boundedness of different observatives do not coincide. The discrepancies originate from the fact that, in shear flow systems, different fields experience algebraic evolutions characterized by different powers of time, while the evolutions for any fields are expressed in a common exponential form for static systems.

ACKNOWLEDGMENTS

The authors thank Professor Wakatani, Professor Mahajan, Professor Berezhiani, and Dr. Furukawa for valuable discussions and suggestions. This work is supported in part by a Grant-in-Aid from the Japanese Ministry of Education, Science, and Culture, Nos. 09308011 and 12780353.

APPENDIX: INHOMOGENEOUS MAGNETIC FIELD

In order to consider the effect of the magnetic shear, let us consider the three-dimensional MHD equation for the evolution of the perturbation fields, which can be written in Cartesian coordinates as

$$\rho_0\left(\frac{\partial \mathbf{v}_1}{\partial t}+\mathbf{v}_0\cdot\nabla\mathbf{v}_1+v_{1x}\frac{\partial \mathbf{v}_0}{\partial x}\right)=\frac{\mathbf{B}_0\cdot\nabla\mathbf{b}}{\mu_0}-\nabla\left(p_0+\frac{\mathbf{B}_0\cdot\mathbf{b}}{\mu_0}\right), \tag{A1}$$

$$\frac{\partial \mathbf{b}}{\partial t}+\mathbf{v}_0\cdot\nabla\mathbf{b}=\mathbf{B}_0\cdot\nabla\mathbf{v}_1+b_x\frac{\partial \mathbf{v}_0}{\partial x}, \tag{A2}$$

where $\mathbf{b}$ denotes the perturbation magnetic field and $\mathbf{v}_0=(0,\sigma x,0)$. Assuming $\mathbf{B}_0=(0,B_{0y}(x),B_{0z}(x))$, we can transform the coordinate as $(x,y,z)\mapsto(x,\eta,\zeta)$ with ζ along the local ambient magnetic field line and η perpendicular to x and ζ. In this coordinate, we have the stationary flow expressed as $(0,v_{0\eta}(x),v_{0\zeta}(x))$. Here, the spatial dependence of the velocity components are

$$v_{0\eta}=\frac{1}{B_0}B_{0z}\sigma x, \quad v_{0\zeta}=\frac{1}{B_0}B_{0y}\sigma x. \tag{A3}$$

If the magnetic fields are homogeneous, the coordinate transform is also spatially homogeneous and these velocity fields are still linear functions with respect to x. Writing the above equations by components in the new coordinates, we have

$$\rho(\partial_t u+v_{0\eta}\partial_\eta u+v_{0\zeta}\partial_\zeta u)=\frac{B_0\partial_\zeta b_x}{\mu_0}-\partial_x\left(p+\frac{B_0 b_\zeta}{\mu_0}\right), \tag{A4}$$

$$\rho\left(\partial_t v+v_{0\eta}\partial_\eta v+v_{0\zeta}\partial_\zeta v+\frac{B_{0z}}{B_0}\sigma u\right)=\frac{B_0\partial_\zeta b_\eta}{\mu_0}-\partial_\eta\left(p+\frac{B_0 b_\zeta}{\mu_0}\right), \tag{A5}$$

$$\rho\left(\partial_t w+v_{0\eta}\partial_\eta w+v_{0\zeta}\partial_\zeta w+\frac{B_{0y}}{B_0}\sigma u\right)=\frac{B_0\partial_\zeta b_\zeta}{\mu_0}-\partial_\zeta\left(p+\frac{B_0 b_\zeta}{\mu_0}\right), \tag{A6}$$

$$\partial_t b_x+v_{0\eta}\partial_\eta b_x+v_{0\zeta}\partial_\zeta b_x=B_0\partial_\zeta u, \tag{A7}$$

$$\partial_t b_\eta+v_{0\eta}\partial_\eta b_\eta+v_{0\zeta}\partial_\zeta b_\eta=B_0\partial_\zeta v+\frac{B_{0z}}{B_0}\sigma b_x, \tag{A8}$$

$$\partial_t b_\zeta+v_{0\eta}\partial_\eta b_\zeta+v_{0\zeta}\partial_\zeta b_\zeta=B_0\partial_\zeta w+\frac{B_{0y}}{B_0}\sigma b_x, \tag{A9}$$

and we can derive for the evolution of the amplitude $\hat{b}_x$, the same equation as previously obtained [Eq. (30)] for $\hat{\psi}$.

If we include the magnetic shear in the stationary condition, the above coordinate transform becomes spatially inhomogeneous. As can be seen from Eqs. (A3), this transform introduces a nonlinear spatial dependence of the background shear flow profile even if this is assumed to be linear in Cartesian coordinates. This fact shows that the introduction of the magnetic shear is essentially equivalent to that of the nonlinear shear flow profile.

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